ON THE LAST KERVAIRE INVARIANT PROBLEM

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Dedicated to Mark Mahowald

Abstract. We prove that the element h_6^2 is a permanent cycle in the Adams spectral sequence. As a result, we establish the existence of smooth framed manifolds with Kervaire invariant one in dimension 126, thereby resolving the final case of the Kervaire invariant problem.

Combining this result with the theorems of Browder, Mahowald–Tangora, Barratt–Jones–Mahowald, and Hill–Hopkins–Ravenel, we conclude that smooth framed manifolds with Kervaire invariant one exist in and only in dimensions $2, 6, 14, 30, 62$, and 126.

Note on this blueprint. This blueprint is adapted from the paper *On the Last Kervaire Invariant Problem* by Weinan Lin, Guozhen Wang, and Zhouli Xu [14]. Compared to the original paper, a "Prerequisites" section has been added, covering foundational concepts (spectral sequences, stable homotopy theory, Adams spectral sequence, etc.) that are not yet formalized in Mathlibv4.28. These additions provide the necessary infrastructure for autoformalization of the full proof in Lean 4.

0. Prerequisites

0.1. **Spectral Sequences.**

0.1.1. *Spectral Sequences.*

Definition 0.1. Let $\mathcal{C}$ be an abelian category. The *nested subspace data* (or *SS data*) for a spectral sequence at a single grading index consists of:

1. An ambient object $V \in \mathcal{C}$.
2. A decreasing family of subobjects $\{Z_r\}_{r\in\mathbb{N}\cup\{\infty\}}$ of V, called *r-cycles*:
$$V = Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_\infty.$$
3. An increasing family of subobjects $\{B_r\}_{r\in\mathbb{N}\cup\{\infty\}}$ of V, called *r-boundaries*:
$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \subseteq B_\infty.$$
4. The containment $B_r \subseteq Z_r$ for all r.
5. Z_∞ is the greatest lower bound of the finite cycles: for any subobject $X \subseteq V$, if $X \subseteq Z_i$ for all $i \in \mathbb{N}$, then $X \subseteq Z_\infty$. Together with the monotonicity of Z, this gives $Z_\infty = \bigcap_{i\in\mathbb{N}} Z_i$.
6. B_∞ is the least upper bound of the finite boundaries: for any subobject $X \subseteq V$, if $B_i \subseteq X$ for all $i \in \mathbb{N}$, then $B_\infty \subseteq X$. Together with the monotonicity of B, this gives $B_\infty = \bigcup_{i\in\mathbb{N}} B_i$.

Together these form the nested chain

$$V \supseteq Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_\infty \supseteq B_\infty \supseteq \cdots \supseteq B_1 \supseteq B_0.$$

The *r-th page* at this index is defined as $E_r = Z_r/B_r$. The *E_∞-page* is $E_\infty = Z_\infty/B_\infty$.

Definition 0.2. A *spectral sequence* in an abelian category $\mathcal{C}$ with index type ι (typically $\mathbb{Z}^n$) consists of:

(1) A starting page index $r_0 \in \mathbb{Z}$.
(2) For each $r \geq r_0$ and index $\mathbf{k} \in \iota$, an object $E_r^{\mathbf{k}} \in \mathcal{C}$ (the *E_r-page* at index $\mathbf{k}$).
(3) A *differential degree function* $\mathbf{d} : \mathbb{Z} \to \iota$, and differentials

$$d_r^{\mathbf{k}} : E_r^{\mathbf{k}} \to E_r^{\mathbf{k}+\mathbf{d}(r)}$$

satisfying $d_r \circ d_r = 0$ componentwise.
(4) For each index $\mathbf{k}$, nested subspace data $(V^{\mathbf{k}}, Z_r^{\mathbf{k}}, B_r^{\mathbf{k}})$ such that $E_r^{\mathbf{k}} = Z_r^{\mathbf{k}}/B_r^{\mathbf{k}}$ for $r \geq r_0$.
(5) An isomorphism $H(E_r, d_r) \cong E_{r+1}$ for each r. By the nesting $B_r \subseteq B_{r+1} \subseteq Z_{r+1} \subseteq Z_r$, the differential d_r has kernel Z_{r+1}/B_r and image B_{r+1}/B_r, so

$$H(E_r, d_r) = \frac{Z_{r+1}/B_r}{B_{r+1}/B_r} \cong Z_{r+1}/B_{r+1} = E_{r+1}$$

by the third isomorphism theorem.

Remark 0.3. All filtrations considered in this project are decreasing.

Definition 0.4. The *E_∞-page* of a spectral sequence E at index $\mathbf{k}$ is given by the nested subspace data: $E_\infty^{\mathbf{k}} = Z_\infty^{\mathbf{k}}/B_\infty^{\mathbf{k}}$. For bounded or degenerate spectral sequences, $E_\infty = E_r$ for all sufficiently large r.

Definition 0.5. A *morphism of spectral sequences* $f : E \to E'$ consists of a family of maps $\varphi_{\mathbf{k}} : V^{\mathbf{k}} \to V'^{\mathbf{k}}$ on the underlying objects at each index, preserving the cycle and boundary subobjects:

$$\varphi_{\mathbf{k}}(Z_r^{\mathbf{k}}) \subseteq Z_r'^{\mathbf{k}}, \qquad \varphi_{\mathbf{k}}(B_r^{\mathbf{k}}) \subseteq B_r'^{\mathbf{k}}$$

for all r, and commuting with differentials on the induced page maps. A morphism induces maps on all pages $E_r \to E'_r$ and on the E_∞-page.

0.1.2. *Convergence of Spectral Sequences.*

Definition 0.6. Let E be a spectral sequence with n-grading and let A be an $(n-1)$-graded object equipped with a decreasing filtration (Definition 0.14). We say E *converges* to A, written $E_{r_0} \Longrightarrow A$, if there exists an isomorphism of n-graded objects

$$E_\infty \cong \operatorname{gr} A,$$

where the n-grading on $\operatorname{gr} A$ is obtained from the $(n-1)$-grading of A together with the filtration index, possibly composed with a fixed reindexing map.

Remark 0.7. This is convergence in the weak sense: we only require an isomorphism between E_∞ and $\operatorname{gr} A$. Conditional convergence and strong convergence (in the sense of Boardman [4]) are not considered in this project.

Definition 0.8. Given a reindexing rule, the *category of converging spectral sequences* has objects (E, A, F, φ) where E is a spectral sequence, A is a filtered graded object, and $\varphi : E_\infty \xrightarrow{\sim} \operatorname{gr} A$ is the convergence isomorphism. A morphism $(E_1, A_1) \to (E_2, A_2)$ consists of:

(1) A morphism of spectral sequences $E_1 \to E_2$ (Definition 0.5), inducing maps on E_∞-pages.
(2) A filtration-preserving map $A_1 \to A_2$.
(3) Compatibility: the induced E_∞-map and the associated graded map commute with the convergence isomorphisms.

Definition 0.9. In the setting of Definition 0.6, the component of the n-grading on E that corresponds to the original $(n-1)$-grading of A is called the *stem* (or *topological degree*).

Definition 0.10. The component of the n-grading on E that corresponds to the filtration on A is called the *filtration degree*.

Definition 0.11. Suppose E converges to A. Let $x \in A$ be an element with filtration at least r, i.e., $x \in F^r A$. Let $\bar{x}$ denote the image of x in $\operatorname{gr}^r A = F^r A / F^{r+1} A$, and let $y \in E_\infty$ be the element corresponding to $\bar{x}$ under the convergence isomorphism. We say y *detects* x.

Proposition 0.12. *An element $x \in F^r A$ is detected by $0 \in E_\infty$ if and only if $x \in F^{r+1} A$.*

Proof. Immediate from the definition of detection and the convergence isomorphism. □

Proposition 0.13. *Two elements $x, x' \in F^r A$ are both detected by the same $y \in E_\infty$ if and only if $x - x' \in F^{r+1} A$.*

Proof. Follows from Proposition 0.12 applied to $x - x'$. □

0.1.3. *Filtered Complex.*

Definition 0.14. A *filtered n-graded abelian group* is an n-graded abelian group A equipped with a decreasing filtration: for each index $\mathbf{k} \in \mathbb{Z}^n$, a family of subgroups

$$\cdots \supseteq F^{s-1} A^{\mathbf{k}} \supseteq F^s A^{\mathbf{k}} \supseteq F^{s+1} A^{\mathbf{k}} \supseteq \cdots$$

Definition 0.15. A *morphism of filtered graded abelian groups* is a graded homomorphism $f : A \to B$ that preserves the filtration: $f(F^s A^{\mathbf{k}}) \subseteq F^s B^{\mathbf{k}}$ for all s and $\mathbf{k}$. Filtered graded abelian groups and their morphisms form a category.

Definition 0.16. A *filtered chain complex* is a chain complex (C_*, d) in the category of filtered graded abelian groups: the differential $d : C_n \to C_{n-1}$ preserves the filtration, i.e., $d(F^s C_n) \subseteq F^s C_{n-1}$ for all s and n.

Proposition 0.17. *A filtered chain complex (C_*, d, F^s) induces a natural filtration on its homology:*

$$F^s H_n(C_*) = \operatorname{im}(H_n(F^s C_*) \to H_n(C_*)) = \ker(H_n(C_*) \to H_n(C_*/F^s C_*)).$$

Definition 0.18. Given a filtered graded abelian group (A, F), the *associated graded* is

$$\operatorname{gr}^s A^{\mathbf{k}} = F^s A^{\mathbf{k}} / F^{s+1} A^{\mathbf{k}}.$$

Definition 0.19. Given a filtered chain complex (C_*, d, F^s), the *associated graded complex* is the bigraded object $\mathrm{gr}^s C_n$ equipped with the induced differential $\bar{d} : \mathrm{gr}^s C_n \to \mathrm{gr}^s C_{n-1}$ (well-defined since d preserves the filtration).

Definition 0.20. A filtered chain complex (C_*, d, F^s) is *exhaustive* if $\bigcup_s F^s C_n = C_n$ for all n.

Definition 0.21. A filtered chain complex (C_*, d, F^s) is *Hausdorff* (or *separated*) if $\bigcap_s F^s C_n = 0$ for all n.

Definition 0.22. A filtered chain complex (C_*, d, F^s) is *bounded* if for each n, there exist $a \le b$ with $F^a C_n = C_n$ and $F^{b+1} C_n = 0$. A bounded filtration is both exhaustive and Hausdorff.

Definition 0.23. Given a filtered chain complex (C_*, d, F^s), we construct an SS-Data structure. The spectral sequence starts at E_0 with $(n+1)$-grading.

1. Define $V^{s,t} = \mathrm{gr}^s C_{s+t}$, the associated graded.
2. Define $Z_r^{s,t}$ as the image in $\mathrm{gr}^s C_{s+t}$ of those $x \in F^s C_{s+t}$ with $dx \in F^{s+r} C_{s+t-1}$.
3. Define $B_r^{s,t}$ as the image in $\mathrm{gr}^s C_{s+t}$ of those elements of the form dy with $y \in F^{s-r} C_{s+t+1}$.

The differential $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$ is induced by d: for $[x] \in Z_r^{s,t}$, define $d_r[x] = [dx] \in E_r^{s+r,t-r+1}$.

Proposition 0.24. *The differential d_r in Definition 0.23 is well-defined and satisfies $d_r \circ d_r = 0$.*

Proof. This is verified by a direct computation in the filtered complex, showing that the composition of two successive page differentials factors through zero. ☐

Theorem 0.25. *The nested subspace data and differentials from Definition 0.23 satisfy the axioms of a spectral sequence. Specifically:*

(1) $B_r \subseteq B_{r+1} \subseteq Z_{r+1} \subseteq Z_r$ *for all* r.
(2) $E_r^{s,t} = Z_r^{s,t} / B_r^{s,t}$.
(3) $H(E_r, d_r) \cong E_{r+1}$.

The spectral sequence starts at $E_0^{s,t} = F^s C_{s+t} / F^{s+1} C_{s+t}$ *with differentials* $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$.

Theorem 0.26. *If the filtration on a filtered chain complex is exhaustive and Hausdorff, then the associated spectral sequence converges to the homology:*

$$E_r^{s,t} \Longrightarrow H_{s+t}(C_*).$$

The convergence isomorphism is $E_\infty^{s,t} \cong \mathrm{gr}^s H_{s+t}(C_*)$ *with respect to the induced filtration from Proposition 0.17.*

Definition 0.27. A *morphism of filtered chain complexes* $f : (C_*, F) \to (D_*, G)$ is a chain map $f : C_* \to D_*$ preserving the filtration: $f(F^s C_n) \subseteq G^s D_n$ for all s, n.

Proposition 0.28. *A morphism of filtered chain complexes induces a morphism of the associated spectral sequences. This construction is functorial.*

0.1.4. *Crossing of Differentials.*

Definition 0.29. Let E be a spectral sequence. A differential $d_r(x) = y$ is called *essential* if $y \neq 0$ on the E_r-page. Equivalently, $d_r(x) = y$ is essential if x is not a boundary and y is not zero in E_r. A differential is *inessential* if $y = 0$ on the E_r-page.

Definition 0.30. A *differential datum* in a spectral sequence E is a triple (r, x, y) where $r \geq r_0$, $x \in E_r^{\mathbf{k}}$, and $y \in E_r^{\mathbf{k}+\mathbf{d}(r)}$ with $d_r(x) = y$. We say the differential datum is *essential* if $y \neq 0$, and denote it $d_r(x) = y$.

Definition 0.31. Let $f : V_1 \to V_2$ be a morphism of converging spectral sequences, and suppose $d_n^f(x) = y$ is an f-extension differential with $x \in E_\infty^s(V_1)$ and $y \in E_\infty^{s+n}(V_2)$.

(1) We say that $d_n^f(x) = y$ *has a crossing that hits filtration* p for some $p \leq s + n$, if there exists $x' \in E_\infty^{s+a}(V_1)$ with $a > 0$ and an essential differential $d_{n-a}^f(x') = y'$ for $0 \neq y' \in E_\infty^{s+a+(n-a)}(V_2)$ such that

$$p \leq \mathrm{Fil}(y') \leq \mathrm{Fil}(y) = s + n.$$

(2) We say that $d_n^f(x) = y$ *has no crossing that hits the range* Fil $\geq p$ if there does not exist such x' and y' with $p \leq \mathrm{Fil}(y') \leq \mathrm{Fil}(y)$.

(3) We say that $d_n^f(x) = y$ *has no crossing* if it has no crossing that hits the range $\mathrm{Fil} \geq \mathrm{Fil}(x) + 1 = s + 1$.

Remark 0.32. The following are equivalent:

- An f-extension $d_n^f(x) = y$ has no crossing that hits the range Fil $\geq p$.
- For any $a > 0$, if there is an f-extension from $x' \in E_\infty^{s+a}(V_1)$ to some nontrivial y', then $\mathrm{Fil}(y') < p$ or $\mathrm{Fil}(y') > \mathrm{Fil}(y)$.

Proposition 0.33. *An f-extension from x to y has no crossing in the range* Fil $\geq p$ *if and only if for all $[x] \in \{x\}$ such that* $\mathrm{Fil}(f_A[x]) \geq p$ *we have $f_A[x] \in \{y\}$. In particular:*

(1) An f-extension from x to y has no crossing if and only if for all $[x] \in \{x\}$ we have $f_A[x] \in \{y\}$.

(2) An f-extension $d_n^f(x) = 0$ has no crossing if and only if for all $[x] \in \{x\}$, $\mathrm{Fil}(f_A[x]) > \mathrm{Fil}(x) + n$.

(3) If $\mathrm{AF}(f) = n$, *then all d_n^f-differentials have no crossing.*

Proof. Only if. Suppose the extension has no crossing in Fil $\geq p$. Assume for contradiction that there exists $[x] \in \{x\}$ with $\mathrm{Fil}(f_A[x]) \geq p$ but $f_A[x] \notin \{y\}$. Let y' be the element of $E_\infty(V_2)$ detecting $f_A[x]$, so $f_A[x] \in \{y'\}$ with $\mathrm{Fil}(y') \geq p$. By Proposition 0.41 there is an f-extension from x to y'. By Proposition 0.42 (2), y' (if $\mathrm{Fil}(y) > \mathrm{Fil}(y')$) or $y - y'$ (if $\mathrm{Fil}(y) = \mathrm{Fil}(y')$) is hit by a shorter d^f-differential from some x' with $\mathrm{Fil}(x') > \mathrm{Fil}(x)$. This gives a crossing that hits Fil $\geq p$, contradicting the assumption.

If. Suppose the extension has a crossing: there exist $x' \in E_\infty^{s+a}(V_1)$ with $a > 0$ and essential $d_{n-a}^f(x') = y'$ with $p \leq \mathrm{Fil}(y') \leq \mathrm{Fil}(y)$. By Proposition 0.41, there exists $[x'] \in \{x'\}$ with $f_A([x']) \in \{y'\}$. Since $\mathrm{Fil}(x') > \mathrm{Fil}(x)$, the element $[x] + [x'] \in \{x\}$ (as $[x']$ lies in higher filtration). Then

$$f_A([x] + [x']) = f_A([x]) + f_A([x']),$$

and this element is detected by y' (if $\mathrm{Fil}(y') < \mathrm{Fil}(y)$) or by $y + y'$ (if $\mathrm{Fil}(y') = \mathrm{Fil}(y)$). In either case $f_A([x]+[x']) \notin \{y\}$ while $\mathrm{Fil}(f_A([x]+[x'])) \geq p$, contradicting the assumption.

Parts (1) and (2) are special cases. Part (3) holds for degree reasons: if $\mathrm{AF}(f) = n$, then f_A raises filtration by at least n, so $\mathrm{Fil}(f_A[x]) \geq s+n$ for all $[x] \in F^s A_1$. $\square$

0.1.5. *Extension Spectral Sequence.*

Definition 0.34. Let $f : V_1 \to V_2$ be a morphism of converging spectral sequences, with underlying map $f_A : A_1 \to A_2$ on the target groups. The *underlying filtered chain complex* of f is the two-term chain complex

$$C_1 = A_1 \xrightarrow{f_A} C_0 = A_2,$$

equipped with the filtrations on A_1 and A_2 inherited from the convergence data: $F^s C_1 = F^s A_1$ and $F^s C_0 = F^s A_2$. Since f_A preserves the filtration, this is a filtered chain complex in the sense of Definition 0.16.

Definition 0.35. The *f-extension spectral sequence* (f-ESS) is the spectral sequence

$$\mathrm{ESS}(f) = \mathrm{FilteredComplex.toSpectralSequence}(C_*(f)),$$

obtained by applying the filtered complex construction (Theorem 0.25) to the underlying filtered chain complex of Definition 0.34. By Theorem 0.26, if the filtrations on A_1 and A_2 are exhaustive and Hausdorff (which holds when they are bounded), then the f-ESS converges to the homology of the underlying complex, i.e., to $\ker(f_A) \oplus \mathrm{coker}(f_A)$.

Proposition 0.36. *The E_0-page of the f-ESS is canonically isomorphic to the direct sum of the E_∞-pages of the source and target:*

$$\mathrm{ESS}(f)_0^s \cong E_\infty^s(V_1) \oplus E_\infty^s(V_2).$$

This follows from $E_0^s = \mathrm{gr}^s C_* = \mathrm{gr}^s A_1 \oplus \mathrm{gr}^s A_2 \cong E_\infty^s(V_1) \oplus E_\infty^s(V_2)$.

Proposition 0.37. *Under the decomposition $\mathrm{ESS}(f)_0^s \cong E_\infty^s(V_1) \oplus E_\infty^s(V_2)$, the differentials of the f-ESS have the form*

$$d_n^{\mathrm{ESS}} : E_\infty^s(V_1) \oplus E_\infty^s(V_2) \to E_\infty^{s+n}(V_1) \oplus E_\infty^{s+n}(V_2),$$

and only the component $E_\infty^s(V_1) \to E_\infty^{s+n}(V_2)$ can be nonzero; the other three components (source-to-source, target-to-source, target-to-target) vanish. This follows from the chain complex structure: the differential of C_* maps $C_1 = A_1$ to $C_0 = A_2$ and is zero in the reverse direction.

Definition 0.38. The *f-extension differential* $d_n^f : E_\infty^s(V_1) \to E_\infty^{s+n}(V_2)$ is the unique nonzero component of the n-th differential of the f-ESS (Proposition 0.37). We say there is an *f-extension* from $x \in E_\infty^s(V_1)$ to $y \in E_\infty^{s+n}(V_2)$ if $d_n^f(x) = y$. The extension is *essential* if y is nontrivial on the E_n-page; otherwise it is *inessential.*

Proposition 0.39. *The zeroth extension differential d_0^f equals the map induced by f on the associated graded:*

$$d_0^f = \mathrm{gr}^s(f_A) : \mathrm{gr}^s A_1 \to \mathrm{gr}^s A_2.$$

Under the identification $\mathrm{gr}^s A_i \cong E_\infty^s(V_i)$, this is the map $E_\infty^s(V_1) \to E_\infty^s(V_2)$ induced by f on E_∞-pages.

Proof. This is the standard formula for the E_0-differential of a filtered complex (Definition 0.23): d_0 is induced by the chain differential on the associated graded. ☐

Notation 0.40. For $x \in E_\infty^s(V_1)$, let $\{x\} \subseteq A_1$ denote the set of elements detected by x:

$$\{x\} = \{\alpha \in F^s A_1 \mid \bar{\alpha} \mapsto x \text{ under } E_\infty^s \cong \mathrm{gr}^s A_1\}.$$

For $[x] \in \{x\}$, we call $[x]$ a *representative* of x. Note that $\{x\}$ is a coset of $F^{s+1}A_1$ in $F^s A_1$.

Proposition 0.41. *An f-extension $d_n^f(x) = y$ holds if and only if there exists $[x] \in \{x\}$ such that $f_A[x] \in \{y\}$.*

Proof. Follows directly from the filtered complex construction: $d_n^f(x) = y$ means there is a lift of x in $F^s A_1$ whose image under f_A lies in $F^{s+n}A_2$ and represents y in $\mathrm{gr}^{s+n} A_2$. ☐

Proposition 0.42. *Consider $f : V_1 \to V_2$, $x \in E_\infty^s(V_1)$, $y \in E_\infty^{s+n}(V_2)$ and $y' \in E_\infty^{s+m}(V_2)$ for $m, n \geq 0$.*

1. *An f-extension from x to y is inessential, i.e., y is trivial on the E_n-page of the f-ESS, if and only if there exists $x' \in E_\infty^{s+a}(V_1)$ for some $0 < a \leq n$ with an essential differential $d_{n-a}^f(x') = y$. Equivalently, there exists $[x'] \in \{x'\} \subseteq A_1$ with $\mathrm{Fil}(x') > \mathrm{Fil}(x)$ such that $f_A[x'] \in \{y\}$.*
2. *Suppose $d_n^f(x) = y$ and $d_m^f(x) = y'$.*
    - (a) *If $m = n$, then $y - y' \in {}^f B_{n-1}^{s+n}(V_2)$: there exists $x' \in E_\infty^{s+a}(V_1)$ for some $0 < a \leq n$ with an essential differential $d_{n-a}^f(x') = y - y'$.*
    - (b) *If $m > n$, then the f-extension from x to y is inessential.*

Proof. Part (1) follows from the definition of the page boundaries in the filtered complex construction. Parts (2a) and (2b) follow from Part (1) and Proposition 0.41. ☐

Proposition 0.43. *Suppose the sequence of converging spectral sequences $V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$ induces an exact sequence on target groups $A_1 \xrightarrow{f_A} A_2 \xrightarrow{g_A} A_3$ that is exact at A_2. Then all permanent d^g-cycles in $E_\infty(V_2)$ are boundaries in the f-ESS.*

Proof. First proof (via chain complexes). Treat the three-term sequence $A_1 \xrightarrow{f_A} A_2 \xrightarrow{g_A} A_3$ as a filtered chain complex. Its associated spectral sequence comprises both the d^f and d^g differentials. By exactness at A_2, the abutment projected to the A_2-component is zero. Therefore every permanent d^g-cycle must be killed by a d^f-differential.

Second proof (via representatives). Let $y \in E_\infty(V_2)$ be a permanent d^g-cycle. By convergence, there exists $[y] \in \{y\}$ with $g_A([y]) = 0$. By exactness, there exists $[x] \in A_1$ with $f_A([x]) = [y]$. Let $x \in E_\infty(V_1)$ detect $[x]$. Then $f_A[x] \in \{y\}$, so by Proposition 0.41 there is an f-extension from x to y. ☐

Unbounded Extension Spectral Sequence. Throughout this subsection, all filtrations are assumed to be *bounded below*: the filtration $F^\bullet A$ satisfies $F^p A = A$ for all sufficiently negative p. Filtrations need not be bounded above. We extend the f-ESS construction to this setting by defining it as the inverse limit of bounded truncations.

Definition 0.44. Let (E, A, F) be a converging spectral sequence with filtration $F^\bullet A$ bounded below. For an integer s, the *s-truncation* of (E, A, F) is the converging spectral sequence $(E^{[\leq s]}, A^{[\leq s]}, F^{[\leq s]})$ defined as follows:

(1) The *truncated spectral sequence*: for each filtration degree p, set
$$V^{[\leq s],p} = \begin{cases} V^p & \text{if } p \leq s, \\ 0 & \text{otherwise,} \end{cases}$$
so that $E_r^{[\leq s],p} = E_r^p$ for $p \leq s$ and $E_r^{[\leq s],p} = 0$ for $p > s$.

(2) The *truncated abutment* is the quotient
$$A^{[\leq s]} = A / F^{s+1} A,$$
equipped with the induced filtration
$$F^{[\leq s],p} A^{[\leq s]} = \begin{cases} \big(F^p A + F^{s+1} A\big) / F^{s+1} A & \text{if } p \leq s, \\ 0 & \text{if } p > s. \end{cases}$$
When $F^p A = A$ (which holds for all sufficiently negative p by the bounded below assumption), this reduces to $F^{[\leq s],p} A^{[\leq s]} = A^{[\leq s]}$.

The truncation $(E^{[\leq s]}, A^{[\leq s]}, F^{[\leq s]})$ is a converging spectral sequence with bounded (hence exhaustive and Hausdorff) filtration.

Proposition 0.45. *The collection $\{(E^{[\leq s]}, A^{[\leq s]})\}_{s \in \mathbb{Z}}$ forms an* inverse system *in the category of converging spectral sequences, indexed by s with order $s \geq s'$. For $s \geq s'$, the transition morphism is the natural projection*
$$A^{[\leq s]} = A / F^{s+1} A \longrightarrow A / F^{s'+1} A = A^{[\leq s']},$$
which is compatible with the truncated filtrations and with the spectral sequence page maps.

Proposition 0.46. *Let (E, A, F) be a converging spectral sequence with filtration bounded below. Then the spectral sequence E is the inverse limit of the spectral sequences of the truncations: at each filtration degree p and page r one has*
$$E_r^p \cong \varprojlim_{s \geq p} E_r^{[\leq s],p}.$$

Proposition 0.47. *Let (E, A, F) be a converging spectral sequence with filtration bounded below. If the filtration on A is* complete *(Definition 0.50), then the converging spectral sequence (E, A) is the inverse limit of the inverse system of truncations in the category of converging spectral sequences:*
$$(E, A) \cong \varprojlim_{s} (E^{[\leq s]}, A^{[\leq s]}).$$

Definition 0.48. Let $f : (E_1, A_1) \to (E_2, A_2)$ be a morphism of converging spectral sequences with filtrations bounded below (not necessarily bounded above). For each integer s, let $f^{[\leq s]} : (E_1^{[\leq s]}, A_1^{[\leq s]}) \to (E_2^{[\leq s]}, A_2^{[\leq s]})$ denote the induced morphism of truncated converging spectral sequences. The *unbounded f-extension spectral sequence* is the spectral sequence (without converging data) defined as the inverse limit in the category of spectral sequences:
$$\mathrm{ESS}^{\mathrm{unbd}}(f) = \varprojlim_{s} \mathrm{ESS}(f^{[\leq s]}),$$

where each $\mathrm{ESS}(f^{[\leq s]})$ is the (bounded) extension spectral sequence of Definition 0.35. The inverse limit is well-defined as a spectral sequence by the stabilization property (Proposition 0.49).

Proposition 0.49. *Let $f : (E_1, A_1) \to (E_2, A_2)$ be a morphism of converging spectral sequences with filtrations bounded below. Fix a filtration degree $p \in \mathbb{Z}$ and a page index $r \geq 1$. Then for all integers $s \geq p + r$, the component of the s-truncated extension spectral sequence at filtration degree p and page r is canonically isomorphic to the corresponding component of the s'-truncated extension spectral sequence for any $s' \geq s$:*

$$\mathrm{ESS}(f^{[\leq s]})_r^{p,*} \;\cong\; \mathrm{ESS}(f^{[\leq s']})_r^{p,*}.$$

In other words, once the truncation level $s \geq p + r$ (so that the truncated ESS retains both the source filtration degree p and the target filtration degree $p + r$), the d_r^f-differential at filtration degree p is fully determined and independent of the truncation level.

Definition 0.50. Let (E, A, F) be a converging spectral sequence with filtration bounded below. The *completion* of A with respect to the filtration F is the inverse limit

$$\widehat{A} = \varprojlim_s A^{[\leq s]} = \varprojlim_s A/F^{s+1}A.$$

There is a natural map $\iota : A \to \widehat{A}$ induced by the projections $A \to A/F^{s+1}A$. The filtration is called *complete* if ι is an isomorphism. The Hausdorff condition (Definition 0.21) ensures ι is injective; completeness further requires ι to be surjective.

Theorem 0.51. *Let $f : (E_1, A_1) \to (E_2, A_2)$ be a morphism of converging spectral sequences with filtrations bounded below. Suppose that the inverse systems $\{A_i^{[\leq s]}\}_{s\in\mathbb{Z}}$ (for $i = 1, 2$) satisfy the* Mittag-Leffler condition*: for each s, the images of the transition maps $A_i^{[\leq s']} \to A_i^{[\leq s]}$ stabilize as $s' \to \infty$. Then the unbounded f-extension spectral sequence $\mathrm{ESS}^{\mathrm{unbd}}(f)$ converges to the bigraded associated-graded homology of the completions:*

$$E_\infty\Big(\mathrm{ESS}^{\mathrm{unbd}}(f)\Big) \cong H_*\Big(\widehat{A}_1 \xrightarrow{\hat{f}_A} \widehat{A}_2\Big),$$

where $\hat{f}_A : \widehat{A}_1 \to \widehat{A}_2$ is the completion of f_A, and the right-hand side denotes the bigraded associated-graded homology of the two-term complex equipped with the filtrations induced on $\widehat{A}_1$ and $\widehat{A}_2$.

0.1.6. *Commutativity of Extension Differentials.*

Theorem 0.52. *Consider a homotopy commutative diagram of converging spectral sequences*

$$\begin{tikzcd} V_1 \arrow[r, "f"] \arrow[d, "p"'] & V_2 \arrow[d, "q"] \\ V_3 \arrow[r, "g"] & V_4 \end{tikzcd}$$

Suppose $m, n, l \geq 0$, $0 < k \leq m + l - n$,

$$x \in E_\infty^s(V_1), \quad y \in E_\infty^{s+n}(V_2), \quad z \in E_\infty^{s+m}(V_3), \quad w \in E_\infty^{s+m+l}(V_4),$$

and

(1) $d^f_n(x) = y$*,*
(2) $d^p_m(x) = z$*,*
(3) the differential in (1) or (2) has no crossing,
(4) $d^g_l(z) = w$*, and has no crossing that hits* $\mathrm{Fil} \geq s + n + k$*,*
(5) $d^q_{k-1}y = 0$*, and has no crossing.*

Then $d^q_{m+l-n}(y) = w$*.*

Proof. First we find a representative $[x] \in \{x\}$ such that $p_A[x] \in \{z\}$ and $f_A[x] \in \{y\}$.

If the differential in (2) has no crossing, by Proposition 0.33 we can pick $[x] \in \{x\}$ such that $p_A[x] \in \{z\}$; by Proposition 0.41, $f_A[x] \in \{y'\}$ for some y', but the no-crossing condition forces $f_A[x] \in \{y\}$. Symmetrically, if the differential in (1) has no crossing, we pick $[x]$ with $f_A[x] \in \{y\}$ and the no-crossing forces $p_A[x] \in \{z\}$.

By (5) and Proposition 0.33 (2), since $d^q_{k-1}y = 0$ has no crossing, $\mathrm{Fil}(q_A f_A[x]) \geq s + n + k$. Since the diagram commutes on target groups, $g_A p_A[x] = q_A f_A[x]$, so $\mathrm{Fil}(g_A p_A[x]) \geq s + n + k$.

Combining with (4) ($d^g_l(z) = w$ has no crossing that hits $\mathrm{Fil} \geq s + n + k$) and $p_A[x] \in \{z\}$, Proposition 0.33 gives $g_A p_A[x] \in \{w\}$.

Therefore $q_A f_A[x] \in \{w\}$, and since $f_A[x] \in \{y\}$, there is a q-extension from y to w by Proposition 0.41. ☐

Corollary 0.53. *In the setting of Theorem 0.52, if condition (4) is strengthened to "*$d^g_l(z) = w$ *has no crossing" (and condition (5) is dropped), then* $d^q_{m+l-n}(y) = w$*.*

Proof. If $d^g_l(z) = w$ has no crossing, then in particular it has no crossing that hits $\mathrm{Fil} \geq s + n + k$ for any k. Taking $k = 1$ and observing that $d^q_0 y = 0$ trivially has no crossing, the result follows from Theorem 0.52. ☐

Corollary 0.54. *Consider a homotopy commutative diagram of converging spectral sequences* $V_1 \xrightarrow{f} V_2 \xrightarrow{q} V_3$ *with* $p = q \circ f$*. If* $d^f_n(x) = y$*,* $d^p_m(x) = z$*, and one of the two extensions has no crossing, then* $d^q_{m-n}(y) = z$*.*

Proof. This is the special case of Corollary 0.53 with $V_3 = V_4$ and $g = \mathrm{id}_{V_3}$: since $d^g_0(z) = z$ and the identity extension always has no crossing, the conditions are satisfied with $l = 0$. ☐

Corollary 0.55. *Consider a homotopy commutative diagram of converging spectral sequences* $V_1 \xrightarrow{p} V_3 \xrightarrow{g} V_4$ *with* $q = g \circ p$*. If* $d^p_m(x) = z$ *and* $d^g_l(z) = w$ *has no crossing, then* $d^q_{m+l}(x) = w$*.*

Proof. This is the special case of Corollary 0.53 with $V_1 = V_2$ and $f = \mathrm{id}_{V_1}$: since $d^f_0(x) = x$ with $n = 0$ and the identity extension has no crossing, the conditions give $d^q_{m+l-0}(y) = d^q_{m+l}(x) = w$. ☐

Corollary 0.56. *If* $g \circ f = 0$ *(as morphisms of converging spectral sequences) and* $d^f_n(x) = y$*, then* y *is a permanent cycle in the* g*-ESS:* $d^g_m(y) = 0$ *for all* $m \geq 0$*.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
V_1 \arrow[r, "f"] \arrow[d] & V_2 \arrow[d, "g"] \\
0 \arrow[r] & V_3
\end{tikzcd}$$

with $p = 0$ and $g \circ f = 0$. All extensions from x to 0 are trivial and have no crossing. By Corollary 0.53, $d^g_m(y) = 0$.

Alternative proof. By Proposition 0.41, there exists $[x] \in \{x\}$ with $f_A[x] \in \{y\}$. Let $[y] = f_A[x]$. Then $g_A([y]) = g_A(f_A([x])) = 0$. Hence y is a permanent d^g-cycle. □

Corollary 0.57. *Consider a homotopy commutative diagram of converging spectral sequences*

$$\begin{tikzcd} V_1 \arrow[r, "f"] \arrow[d, "p"'] & V_2 \arrow[d, "q"] \\ V_3 \arrow[r, "g"] & V_4 \end{tikzcd}$$

Assume ${}^pE_0 = {}^pE_r$ and ${}^qE_0 = {}^qE_r$ for some $r \geq 0$ (i.e., the p-ESS and q-ESS have stationary pages up to r). Then (d^p_r, d^q_r) induces a map from the f-ESS to the g-ESS.

Proof. Since ${}^pE_0 = {}^pE_r$ and ${}^qE_0 = {}^qE_r$, the differentials d^p_r and d^q_r have no crossings. By Corollary 0.53, $d^g_0 \circ d^p_r = d^q_r \circ d^f_0$ and therefore (d^p_r, d^q_r) induces a map from ${}^fE_1$ to ${}^gE_1$. Inductively, applying Corollary 0.53 shows that (d^p_r, d^q_r) induces a map from ${}^fE_n$ to ${}^gE_n$ and $d^g_n \circ d^p_r = d^q_r \circ d^f_n$ for all n. □

0.2. **Stable Homotopy Theory.**

0.2.1. *Stable Homotopy Theory.*

Definition 0.58. A *triangulated category* is an additive category $\mathcal{T}$ equipped with:

1. An additive autoequivalence $\Sigma : \mathcal{T} \to \mathcal{T}$, called the *shift* (or *suspension*) functor.
2. A class of *distinguished triangles* $X \to Y \to Z \to \Sigma X$, satisfying the axioms (TR1)–(TR4):
   - (TR1) Every morphism $f : X \to Y$ extends to a distinguished triangle $X \xrightarrow{f} Y \to Z \to \Sigma X$. The triangle $X \xrightarrow{\text{id}} X \to 0 \to \Sigma X$ is distinguished.
   - (TR2) A triangle $X \to Y \to Z \to \Sigma X$ is distinguished if and only if $Y \to Z \to \Sigma X \to \Sigma Y$ is distinguished.
   - (TR3) Given a morphism of the first two terms of two distinguished triangles, there exists a (not necessarily unique) morphism of the third terms making the diagram commute.
   - (TR4) The octahedral axiom.

Definition 0.59. In a triangulated category, a distinguished triangle $X \xrightarrow{f} Y \to C_f \to \Sigma X$ is called a *cofiber sequence*, and C_f is the *cofiber* (or *cone*) of f. By (TR1), every morphism admits a cofiber sequence.

Definition 0.60. A *functorial cofiber* on a triangulated category $\mathcal{T}$ is an assignment that to each morphism $f : X \to Y$ in $\mathcal{T}$ associates a distinguished triangle $X \xrightarrow{f} Y \to Cf \to \Sigma X$ that is functorial: given a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

there is an induced map $C\alpha\beta : Cf \to Cf'$ making the diagram of distinguished triangles commute.

Proposition 0.61. *For a distinguished triangle $X \to Y \to Z \to \Sigma X$ and any object W, the long exact sequence on hom-sets*

$$\cdots \to [W, X] \to [W, Y] \to [W, Z] \to [W, \Sigma X] \to \cdots$$

holds (covariant), and dually for the contravariant case.

Definition 0.62. A *closed symmetric monoidal category* is a category $\mathcal{C}$ equipped with:

1. A unit object S.
2. A bifunctor $(X, Y) \mapsto X \wedge Y$ (the *smash product* or *tensor product*) from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$, which is associative and commutative up to coherent natural isomorphism, with $S \wedge X \cong X$ up to coherent natural isomorphism.
3. Function objects $F(X, Y)$, functorial contravariantly in X and covariantly in Y, satisfying the tensor-hom adjunction
$$[X, F(Y, Z)] \cong [X \wedge Y, Z]$$
naturally in all three variables.

Definition 0.63. A *closed symmetric tensor triangulated category* is a triangulated category $\mathcal{C}$ equipped with a closed symmetric monoidal structure (Definition 0.62) that is *compatible* with the triangulation:

1. The smash product preserves suspensions: there is a natural equivalence $e_{X,Y} : \Sigma X \wedge Y \xrightarrow{\sim} \Sigma(X \wedge Y)$, compatible with the unit and associativity isomorphisms.
2. The smash product is exact: if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is an exact triangle, then for any object W,
$$X \wedge W \xrightarrow{f \wedge 1} Y \wedge W \xrightarrow{g \wedge 1} Z \wedge W \xrightarrow{h \wedge 1} \Sigma(X \wedge W)$$
is exact.
3. The functor $F(X, Y)$ is exact in Y; it is exact in X up to sign.
4. The smash product interacts with suspension in a graded-commutative manner: the twist map $T : S^r \wedge S^s \to S^s \wedge S^r$ and the equivalence $S^r \wedge S^s \simeq S^{r+s}$ satisfy $T = (-1)^{rs}$.

This definition follows Hovey–Palmieri–Strickland [9], Definition A.2.1.

Definition 0.64. A *closed symmetric tensor triangulated category with functorial cofiber* is a closed symmetric tensor triangulated category $\mathcal{T}$ (Definition 0.63) equipped with:

1. A functorial cofiber (Definition 0.60).
2. A *tensor-cofiber exchange*: for any morphism $f : X \to Y$ and any object W, a natural isomorphism
$$C(f \wedge W) \cong C(f) \wedge W,$$
where $f \wedge W : X \wedge W \to Y \wedge W$ denotes the right-tensoring of f with W (i.e., $f \otimes \mathrm{id}_W$), compatible with the triangulated structure.

Definition 0.65. The *stable homotopy category* $\mathcal{S}$ is a closed symmetric tensor triangulated category with functorial cofiber (Definition 0.64). Its objects are called *spectra.*

Remark 0.66. In this project, $\mathcal{S}$ refers to the homotopy category of spectra (in the sense of stable homotopy theory). It can be constructed as the homotopy category of a stable model category, but we take it as given and work axiomatically with its triangulated and closed symmetric monoidal structure.

Definition 0.67. Given morphisms $f : X \to Y$ and $g : X \to Z$ in $\mathcal{S}$, the *pushout* of the diagram $Y \xleftarrow{f} X \xrightarrow{g} Z$ is the cofiber $C(f, g)$ of the canonical morphism $(f, g) : X \to Y \vee Z$, where $Y \vee Z$ denotes the coproduct (wedge sum) in $\mathcal{S}$.

Definition 0.68. The *sphere spectrum* $\mathbb{S} \in \mathcal{S}$ is the unit object for the smash product.

Definition 0.69. For an integer $n \in \mathbb{Z}$, define

$$S^n = \Sigma^n \mathbb{S},$$

the n-fold suspension of the sphere spectrum.

Definition 0.70. For a spectrum $X \in \mathcal{S}$, the *n-th homotopy group* of X is

$$\pi_n X = [S^n, X] = \mathrm{Hom}_{\mathcal{S}}(S^n, X).$$

Proposition 0.71. *For any spectrum X, the collection $\pi_* X = \{\pi_n X\}_{n\in\mathbb{Z}}$ is a graded abelian group.*

Definition 0.72. The *smash product* $\wedge : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ is the symmetric monoidal product from the closed symmetric monoidal structure of $\mathcal{S}$ (Definition 0.63), with unit $\mathbb{S}$. It satisfies $S^m \wedge S^n \simeq S^{m+n}$ and induces a pairing on homotopy groups:

$$\pi_m X \otimes \pi_n Y \to \pi_{m+n}(X \wedge Y).$$

Definition 0.73. For spectra $X, Y \in \mathcal{S}$, the *mapping spectrum* $F(X, Y)$ is the internal hom (function object) from the closed symmetric monoidal structure (Definition 0.63), adjoint to the smash product:

$$[X \wedge Y, Z] \cong [X, F(Y, Z)].$$

In particular, $\pi_n F(X, Y) \cong [\Sigma^n X, Y]$.

Proposition 0.74. *Let $X \xrightarrow{f} Y \to C_f \to \Sigma X$ be a cofiber sequence. Then there is a long exact sequence of homotopy groups:*

$$\cdots \to \pi_{n+1}(C_f) \to \pi_n(X) \xrightarrow{f_*} \pi_n(Y) \to \pi_n(C_f) \to \pi_{n-1}(X) \to \cdots$$

This follows from Proposition 0.61 and the representability $\pi_n(-) = [S^n, -]$.

Proof. Follows from the long exact Hom sequence applied to representable functors. □

0.2.2. *Cohomology.*

Definition 0.75. The *Eilenberg–MacLane spectrum* $\mathrm{H}\mathbb{F}_2$ is the spectrum representing mod 2 ordinary cohomology. It is a commutative ring spectrum in $\mathcal{S}$.

Axiom 0.76. *The homotopy groups of the Eilenberg–MacLane spectrum are*

$$\pi_n(\mathrm{H}\mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Definition 0.77. The *mod 2 cohomology* of a spectrum X is

$$H^n(X; \mathbb{F}_2) = [\Sigma^n X, \mathrm{H}\mathbb{F}_2].$$

Definition 0.78. The *mod 2 homology* of a spectrum X is

$$H_n(X; \mathbb{F}_2) = \pi_n(\mathrm{H}\mathbb{F}_2 \wedge X).$$

Axiom 0.79. *(Universal Coefficient Theorem) For any spectrum X and $n \in \mathbb{Z}$, the mod 2 cohomology is canonically isomorphic to the $\mathbb{F}_2$-linear dual of the mod 2 homology:*

$$H^n(X; \mathbb{F}_2) \cong \mathrm{Hom}_{\mathbb{F}_2}(H_n(X; \mathbb{F}_2),\ \mathbb{F}_2).$$

0.2.3. *Adams Spectral Sequence.*

Construction.

Axiom 0.80. *The* $\mathrm{H}\mathbb{F}_2$-Adams spectral sequence *is a functor from $\mathcal{S}^{op} \times \mathcal{S}$ to the category of bigraded spectral sequences. For spectra X and Y, it produces a spectral sequence $\{E_r^{s,t}(X, Y), d_r\}$ with $r \geq 2$ and differentials*

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}.$$

A map $f : X' \to X$ induces maps $f^* : E_r^{s,t}(X, Y) \to E_r^{s,t}(X', Y)$, and a map $g : Y \to Y'$ induces maps $g_* : E_r^{s,t}(X, Y) \to E_r^{s,t}(X, Y')$, both compatible with differentials. When $X = \mathbb{S}$, we write $E_r^{s,t}(Y) = E_r^{s,t}(\mathbb{S}, Y)$.

Axiom 0.81. *All elements in the E_2-page of the mod 2 Adams spectral sequence have order 2, i.e., $2x = 0$ for all $x \in E_2^{s,t}(X)$. This holds because $E_2^{s,t}(X)$ is an $\mathbb{F}_2$-vector space.*

Definition 0.82. The full subcategory $\mathcal{S}^{fin} \subset \mathcal{S}$ of *finite spectra* is the smallest full subcategory containing the sphere spectrum $\mathbb{S}$ and closed under suspension, desuspension, and taking cofibers.

Theorem 0.83. *$Z \in \mathcal{S}^{fin}$ if and only if Z can be obtained from $\mathbb{S}$ by finitely many operations of the following two types:*

(1) suspension or desuspension,
(2) taking the cofiber of a map from a sphere spectrum S^n.

Connectivity.

Definition 0.84. A spectrum X is *n-connected* if $\pi_i(X) = 0$ for all $i \leq n$.

Axiom 0.85. *The sphere spectrum $\mathbb{S}$ is (-1)-connected.*

Definition 0.86. A spectrum X is *bounded below* if it is n-connected for some $n \in \mathbb{Z}$.

Axiom 0.87. *If X is n-connected, then $H_i(X; \mathbb{F}_2) = 0$ for all $i \leq n$.*

Finite type.

Definition 0.88. A spectrum X is of *finite type* if for each n, there exists a finite spectrum $X_n \in \mathcal{S}^{fin}$ and a map $X_n \to X$ whose cofiber is n-connected.

Axiom 0.89. *A spectrum X is of finite type if and only if:*

1. *X is bounded below, and*
2. *for each i, the homotopy group $\pi_i(X)$ is finitely generated as an abelian group.*

Axiom 0.90. *If X is of finite type, then $H_i(X;\mathbb{F}_2)$ is a finite-dimensional $\mathbb{F}_2$-vector space for each i.*

Convergence and boundedness. In this paragraph, assume $X \in \mathcal{S}^{fin}$ and Y is of finite type.

Definition 0.91. The Adams spectral sequence induces a natural decreasing filtration on π_*X, called the *Adams filtration.* An element $\alpha \in \pi_nX$ has *Adams filtration* $\geq s$ if it lies in $F^s\pi_nX$.

Definition 0.92. The Adams filtration extends to maps between spectra. For $X \in \mathcal{S}^{fin}$ and Y of finite type, a map $f \in [X,Y]$ has *Adams filtration* $\mathrm{AF}(f) \geq k$ if f factors as a composition

$$X \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{k-1}} X_{k-1} \xrightarrow{f_k} Y$$

where each f_i induces the zero map on mod 2 homology: $(f_i)_* = 0 : H_*(X_{i-1};\mathbb{F}_2) \to H_*(X_i;\mathbb{F}_2)$. The *Adams filtration* $\mathrm{AF}(f)$ is the largest such k.

Proposition 0.93. *A map $f : X \to Y$ with $\mathrm{AF}(f) = k$ maps $F^s\pi_*X$ into $F^{s+k}\pi_*Y$ for all s.*

Corollary 0.94. *If $\mathrm{AF}(f) = k$, then $d_i^f = 0$ for $i < k$ in the f-extension spectral sequence.*

Proof. By Proposition 0.41, $d_i^f(x) = y$ requires $[x] \in \{x\}$ with $f_A[x] \in \{y\}$, which forces $\mathrm{Fil}(f_A[x]) \geq s+i$. But $\mathrm{AF}(f) = k$ means f_A raises filtration by at least k, so $\mathrm{Fil}(f_A[x]) \geq s+k > s+i$ for $i < k$, hence $y = 0$. ☐

Definition 0.95. A map $f \in [X,Y]$ has *Adams filtration* ∞ if $\mathrm{AF}(f) \geq n$ for all n.

Theorem 0.96. *If f has odd order in $[X,Y]$, then f has Adams filtration ∞.*

Axiom 0.97. *If f does not have odd order, then f does not have Adams filtration ∞.*

Axiom 0.98. *For $X \in \mathcal{S}^{fin}$ and Y of finite type, the Adams spectral sequence $Adams(X,Y)$ weakly converges to $[X,Y]$: there is a natural isomorphism between the graded pieces of the Adams filtration on $[X,Y]$ and the E_∞-page of $Adams(X,Y)$.*

Theorem 0.99. *The quotient $[X,Y]/\{$odd-order elements$\}$ carries an induced Adams filtration that is separated (Hausdorff).*

Axiom 0.100. *Suppose X is of finite type. Then:*

1. *If additionally $H_i(X;\mathbb{Q}) = 0$ for some i, then the Adams filtration on $\pi_i(X)/\{$odd-order elements$\}$ is bounded.*
2. *All subquotients of the Adams filtration on $\pi_i(X)/\{$odd-order elements$\}$ are finite groups.*

0.3. $\mathrm{H}\mathbb{F}_2$-Synthetic Spectra.

0.3.1. *Synthetic Spectra.*

Definition 0.101. The category $h\mathrm{Syn}$ of $\mathrm{H}\mathbb{F}_2$-*synthetic spectra* is a closed symmetric monoidal triangulated category with functorial cofiber (it arises as the homotopy category of a stable model category $\mathrm{Syn}_{\mathrm{H}\mathbb{F}_2}$). We denote its tensor product by $\otimes$ and its unit object by $\mathbf{1}$.

Definition 0.102. The category $h\mathrm{Syn}$ admits a *bigraded suspension* $\Sigma^{m,n}$ for each $(m,n) \in \mathbb{Z}^2$, which is an autoequivalence of $h\mathrm{Syn}$ satisfying

$$\Sigma^{m_1,n_1}\Sigma^{m_2,n_2} \cong \Sigma^{m_1+m_2,n_1+n_2}.$$

The functor $\Sigma^{1,0}$ is identified with the triangulated suspension functor.

Axiom 0.103. *The bigraded suspension commutes with the tensor product: for all* $(m,n) \in \mathbb{Z}^2$ *and objects* X, Y *of* $h\mathrm{Syn}$*,*

$$\Sigma^{m,n}(X \otimes Y) \cong \Sigma^{m,n} X \otimes Y.$$

Definition 0.104. There exists a natural transformation

$$\lambda : \Sigma^{0,-1} \to \mathrm{Id}$$

in $h\mathrm{Syn}$. For a synthetic spectrum X, λ_X denotes the component $\Sigma^{0,-1}X \to X$.

Definition 0.105. For $n \geq 0$, define $\lambda_X^n : \Sigma^{0,-n}X \to X$ by induction:

1. $\lambda_X^0 = \mathrm{id}_X : X \to X$.
2. $\lambda_X^{n+1} = \lambda_X \circ \Sigma^{0,-1}(\lambda_X^n) : \Sigma^{0,-n-1}X \to \Sigma^{0,-1}X \to X$, using the composition isomorphism $\Sigma^{0,-n-1} \cong \Sigma^{0,-1}\Sigma^{0,-n}$.

We define X/λ^n as the cofiber of λ_X^n. In particular, $X/\lambda^0 = 0$ and $X/\lambda^1 = X/\lambda$. The cofiber construction gives a distinguished triangle

$$\Sigma^{0,-n}X \xrightarrow{\lambda_X^n} X \to X/\lambda^n \to \Sigma^{1,-n}X.$$

Proposition 0.106. *The homotopy category* $h\mathrm{Syn}$ *is enriched over* $\mathbb{Z}[\lambda]$*-modules.*

Proof. The argument proceeds in three steps:

1. Since $h\mathrm{Syn}$ is preadditive, it is enriched over $\mathbb{Z}$-modules.
2. The natural transformation λ induces an action on hom-sets: for $f : X \to Y$, define $\lambda \cdot f = \lambda_Y \circ \Sigma^{0,-1}(f)$. By naturality of λ, this agrees with $f \circ \lambda_X$ (up to the identification $\Sigma^{0,-1}X \to X$), and is compatible with composition.
3. The polynomial ring $\mathbb{Z}[\lambda]$ is the free $\mathbb{Z}$-algebra on one generator. By its universal property, the $\mathbb{Z}$-linear action of λ on each $\mathrm{Hom}(X,Y)$ extends uniquely to a $\mathbb{Z}[\lambda]$-module structure.

□

0.3.2. *Synthetic Spheres.*

Definition 0.107. The *synthetic sphere* $S^{0,0}$ is defined as the monoidal unit $\mathbf{1}$ of $h\mathrm{Syn}$. The *bigraded spheres* are

$$S^{m,n} = \Sigma^{m,n}S^{0,0} = \Sigma^{m,n}\mathbf{1}.$$

Proposition 0.108. *For any synthetic spectrum* X*,*

$$\Sigma^{m,n}X \cong S^{m,n} \otimes X.$$

Proof. By Axiom 0.103 and the unit isomorphism $\mathbf{1} \otimes X \cong X$: $\Sigma^{m,n} X \cong \Sigma^{m,n}(\mathbf{1} \otimes X) \cong \Sigma^{m,n}\mathbf{1} \otimes X = S^{m,n} \otimes X$. ☐

Definition 0.109. The *synthetic homotopy groups* of a synthetic spectrum X are

$$\pi_{m,n}(X) = [S^{m,n}, X].$$

These form a bigraded abelian group with a natural $\mathbb{Z}[\lambda]$-module structure (Proposition 0.106).

Proposition 0.110.

$$\pi_{m,n}(X) \cong \pi_{m+k,n+l}(\Sigma^{k,l} X).$$

Proof. By the composition axiom $\Sigma^{m,n}\Sigma^{k,l} \cong \Sigma^{m+k,n+l}$, the functor $\Sigma^{k,l}$ is an autoequivalence and induces a bijection $[S^{m,n}, X] \cong [\Sigma^{k,l} S^{m,n}, \Sigma^{k,l} X] \cong [S^{m+k,n+l}, \Sigma^{k,l} X]$. ☐

Definition 0.111. The *λ-action* on synthetic homotopy groups is the map

$$\lambda : \pi_{m,n}(X) \to \pi_{m,n-1}(X)$$

defined as follows: given $f : S^{m,n} \to X$, the element $\lambda \cdot f \in \pi_{m,n-1}(X)$ is the composite

$$S^{m,n-1} \xrightarrow[\sim]{\text{biShift}} \Sigma^{0,-1} S^{m,n} \xrightarrow{\lambda_{S^{m,n}}} S^{m,n} \xrightarrow{f} X,$$

where the first arrow is the biShift composition isomorphism $S^{m,n-1} = \Sigma^{m,n-1}\mathbf{1} \cong \Sigma^{0,-1}\Sigma^{m,n}\mathbf{1} = \Sigma^{0,-1} S^{m,n}$, and $\lambda_{S^{m,n}} : \Sigma^{0,-1} S^{m,n} \to S^{m,n}$ is the component of λ at $S^{m,n}$.

0.3.3. *Synthetic Adams Spectral Sequence.*

Axiom 0.112. *The* synthetic Adams spectral sequence *is a functor from* $h\mathrm{Syn}$ *to the category of 3-graded spectral sequences. For a synthetic spectrum* $X \in h\mathrm{Syn}$*, it produces a spectral sequence* ${}^{\mathrm{syn}}E_r^{s,t,w}(X)$ *with differentials*

$$d_r : {}^{\mathrm{syn}}E_r^{s,t,w} \to {}^{\mathrm{syn}}E_r^{s+r,t+r-1,w}.$$

For a finite spectrum $X \in \mathcal{S}^{\mathrm{fin}}$*, the synthetic Adams spectral sequence for* νX *converges to* $\pi_{*,*}(\nu X)$*.*

Remark 0.113. The synthetic Adams spectral sequence is a $\mathbb{Z}[\lambda]$-module spectral sequence, with λ in tridegree $(0, 0, -1)$.

0.3.4. *The ν Functor.*

Definition 0.114. There exists an additive functor

$$\nu : h\mathcal{S} \to h\mathrm{Syn}$$

from the homotopy category of spectra to the homotopy category of synthetic spectra.

Axiom 0.115. *The functor* ν *intertwines suspension with bigraded suspension:*

$$\nu(\Sigma X) \cong \Sigma^{1,1}\nu(X).$$

The natural comparison map $\Sigma(\nu X) \to \nu(\Sigma X)$ *is induced by* λ*.*

Theorem 0.116. *For all* $n \in \mathbb{Z}$ *and spectra* X*,*

$$\nu(\Sigma^n X) \cong \Sigma^{n,n}\nu(X).$$

Proof. By induction on n. The base case $n = 1$ is Axiom 0.115. The inductive step follows from the biShift composition isomorphism $\Sigma^{n,n}\Sigma^{1,1} \cong \Sigma^{n+1,n+1}$. $\square$

Axiom 0.117. *Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence of spectra such that*

$$0 \to \mathrm{H}\mathbb{F}_{2*}X \xrightarrow{\mathrm{H}\mathbb{F}_{2*}f} \mathrm{H}\mathbb{F}_{2*}Y \xrightarrow{\mathrm{H}\mathbb{F}_{2*}g} \mathrm{H}\mathbb{F}_{2*}Z \to 0$$

is a short exact sequence of $\mathrm{H}\mathbb{F}_2$-homology. Then $\nu X \xrightarrow{\nu f} \nu Y \xrightarrow{\nu g} \nu Z$ is a distinguished triangle in $h\mathrm{Syn}$.

Remark 0.118. The cohomological variant also holds: if $X \to Y \to Z$ is a cofiber sequence with

$$0 \to H^*(Z;\mathbb{F}_2) \to H^*(Y;\mathbb{F}_2) \to H^*(X;\mathbb{F}_2) \to 0$$

short exact in mod 2 cohomology, then ν sends it to a distinguished triangle. This follows from Axiom 0.117 and the universal coefficient theorem (Axiom 0.79).

0.3.5. *Synthetic Rigidity.*

Axiom 0.119. *Let $X \in \mathcal{S}^{\mathrm{fin}}$ be a finite spectrum. The synthetic Adams spectral sequence for νX has E_2-page*

$${}^{\mathrm{syn}}E_2^{*,*,*}(\nu X) \cong E_2^{*,*}(X) \otimes \mathbb{F}_2[\lambda],$$

where an element in $E_2^{s,t}(X)$ has tridegree (s,t,t) and λ has tridegree $(0,0,-1)$. Given a classical Adams differential $d_r^{\mathrm{cl}}(x) = y$, the corresponding synthetic differential is $d_r(x) = \lambda^{r-1}y$, which is λ-linear, and all synthetic Adams differentials arise in this way.

Axiom 0.120. *Let $X \in \mathcal{S}^{\mathrm{fin}}$ be a finite spectrum. The synthetic Adams spectral sequence for νX is isomorphic to the λ-Bockstein spectral sequence:*

1. *$E_2^{s,t}(X) \cong \pi_{t-s,t}(\nu X/\lambda)$.*
2. *If there is a classical Adams differential $d_r x = y$ for $x \in E_2^{s,t}(X) \cong \pi_{t-s,t}(\nu X/\lambda)$, then x admits a lift to $\pi_{t-s,t}(\nu X/\lambda^{r-1})$ whose image under the Bockstein $\nu X/\lambda^{r-1} \to \Sigma^{1,-r+1}\nu X/\lambda$ equals $d_r(x)$.*

Proposition 0.121. *Let $X \in \mathcal{S}^{\mathrm{fin}}$ be a finite spectrum. The E_∞-page of the synthetic Adams spectral sequence for νX is*

$$E_\infty^{s,t,w}(\nu X) \cong \begin{cases} Z_\infty^{s,t}(X)/B_{1+t-w}^{s,t}(X) & \text{if } t \geq w, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Follows from the rigidity degeneration axiom by unwinding the associated graded structure. $\square$

Proposition 0.122. *Let $X \in \mathcal{S}^{\mathrm{fin}}$ be a finite spectrum. For $r \geq 2$, the E_∞-page of the synthetic Adams spectral sequence for $\nu X/\lambda^r$ is*

$$E_\infty^{s,t,w}(\nu X/\lambda^r) \cong \begin{cases} Z_{r-t+w}^{s,t}(X)/B_{1+t-w}^{s,t}(X) & \text{if } 0 \leq t-w < r, \\ 0 & \text{otherwise.} \end{cases}$$

0.3.6. *Synthetic Lift.*

Axiom 0.123. *Let $X, Y \in \mathcal{S}^{\text{fin}}$ be finite spectra. If a map $f : X \to Y$ has Adams filtration $\mathrm{AF}(f) = k$, then there exists a factorization*

$$\begin{tikzcd}
 & \Sigma^{0,-k}\nu Y \arrow[d, "\lambda^k"] \\
\nu X \arrow[r, "\nu f"] & \nu Y
\end{tikzcd}$$

with a dashed lift $\Sigma^{0,-k}\tilde{f} : \nu X \dashrightarrow \Sigma^{0,-k}\nu Y$ such that $\lambda^k \circ \Sigma^{0,-k}\tilde{f} = \nu f$, where $\tilde{f} : \Sigma^{0,k}\nu X \to \nu Y$ is called a synthetic lift *of f.*

Axiom 0.124. *Let $X, Y \in \mathcal{S}^{\text{fin}}$ be finite spectra and let $f : X \to Y$ be a map that induces the zero map on $\mathrm{H}\mathbb{F}_2$-homology: $H_*(f; \mathbb{F}_2) = 0$. Let*

$$W \xrightarrow{i} X \xrightarrow{f} Y \xrightarrow{h} \Sigma W$$

be a functorial distinguished triangle (so $W = \Sigma^{-1}C(f)$ is the desuspended cofiber). By the ν-functor applied to this triangle, there exists a distinguished triangle of synthetic spectra

$$\nu(\Sigma^{-1}Y) \xrightarrow{g} \nu W \xrightarrow{\nu i} \nu X,$$

where g denotes the boundary map. Then:

$$\lambda g = \nu f,$$

i.e., the boundary map g composed with λ equals the synthetic image of f.

Proposition 0.125. *Let $X, Y, Z \in \mathcal{S}^{\text{fin}}$ be finite spectra. Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a distinguished triangle of spectra with $\mathrm{AF}(h) > 0$, and consequently a short exact sequence on $\mathrm{H}\mathbb{F}_2$-homology*

$$0 \to H_*X \xrightarrow{H_*f} H_*Y \xrightarrow{H_*g} H_*Z \to 0.$$

Then there exists a distinguished triangle of synthetic spectra

$$\nu X \xrightarrow{\nu f} \nu Y \xrightarrow{\nu g} \nu Z \xrightarrow{\Sigma^{0,-1}\hat{h}} \Sigma^{0,-1}\nu\Sigma X = \Sigma^{1,0}\nu X$$

such that $\nu h = \lambda\hat{h}$. The last relation $\nu h = \lambda\hat{h}$ follows from Axiom 0.124.

Notation 0.126. For a map $f : X \to Y$ which is part of a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Cf \to \Sigma X$, define

$$e(f) = \begin{cases} 0 & \text{if } \mathrm{AF}(f) = 0, \\ 1 & \text{if } \mathrm{AF}(f) > 0. \end{cases}$$

When $\mathrm{AF}(f) = 0$, we also denote νf by $\hat{f}$. In both cases, we have $\hat{f} : \Sigma^{0,e(f)}\nu X \to \nu Y$ and $\nu f = \lambda^{e(f)}\hat{f}$. Furthermore, $C\hat{f} \simeq \Sigma^{0,-e(g)}\nu Cf$ (by Axiom 0.124).

Proposition 0.127. *Let $X, Y, Z \in \mathcal{S}^{\text{fin}}$ be finite spectra. Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a distinguished triangle with $e(f) + e(g) + e(h) = 1$. Then there is a distinguished triangle of synthetic spectra*

$$\nu X \xrightarrow{\hat{f}} \Sigma^{0,-e(f)}\nu Y \xrightarrow{\hat{g}} \Sigma^{0,-e(f)-e(g)}\nu Z \xrightarrow{\hat{h}} \Sigma^{0,-1}\nu\Sigma X.$$

The construction uses Axiom 0.124 to establish the λ-relations between the classical and synthetic connecting maps.

4. Synthetic Extensions

For any map $f : X \to Y$ of synthetic spectra, we have the induced morphism of converging spectral sequences. The resulting extension spectral sequence is denoted by the ESS of f.

4.1. **Lambda extensions.**

Notation 4.1. There are certain special maps between synthetic spectra that we want to consider. For any $n < m \leq \infty$ and a spectrum X, we have the following distinguished triangles of $\mathrm{H}\mathbb{F}_2$-synthetic spectra

$$\Sigma^{0,-n}\nu X/\lambda^{m-n} \xrightarrow{\lambda^n} \nu X/\lambda^m \xrightarrow{\rho_{n,m}} \nu X/\lambda^n \xrightarrow{\delta_{n,m}} \Sigma^{1,-n}\nu X/\lambda^{m-n}.$$

We simply write $\rho = \rho_{n,m}$, $\delta = \delta_{n,m}$ by abuse of notation if n, m is understood in the context. When $m = \infty$, this sequence is interpreted as

$$\Sigma^{0,-n}\nu X \xrightarrow{\lambda^n} \nu X \xrightarrow{\rho} \nu X/\lambda^n \xrightarrow{\delta} \Sigma^{1,-n}\nu X.$$

Proposition 4.2. *The only nonzero differentials in the extension spectral sequences for the maps λ^n and ρ from Notation 4.1 are the d_0's:*

$$d_0^{\lambda^n} = \lambda^n \text{ and } d_0^{\rho} = \rho.$$

As a result, these d_0's have no crossings.

Proof. First, we show that the the following sequence is exact in the middle:

$$E_\infty^{s,t,w}(\Sigma^{0,-n}\nu X/\lambda^{m-n}) \xrightarrow{\lambda^n} E_\infty^{s,t,w}(\nu X/\lambda^m) \xrightarrow{\rho} E_\infty^{s,t,w}(\nu X/\lambda^n). \tag{4.3}$$

We apply Proposition 0.121 and prove this case by case.

When $t - w < 0$ or $t - w \geq m$, the middle group $E_\infty^{s,t,w}(\nu X/\lambda^m) = 0$, so the sequence is exact in the middle.

When $0 \leq t - w < n$, the sequence is isomorphic to

$$0 \to Z_{m-t+w}^{s,t}(X)/B_{1+t-w}^{s,t}(X) \to Z_{n-t+w}^{s,t}(X)/B_{1+t-w}^{s,t}(X)$$

which is exact in the middle because the second map is clearly injective.

When $n \leq t - w < m$, the sequence is isomorphic to

$$Z_{m-t+w}^{s,t}(X)/B_{1+t-w-n}^{s,t}(X) \to Z_{m-t+w}^{s,t}(X)/B_{1+t-w}^{s,t}(X) \to 0$$

which is exact in the middle because the first map is clearly surjective.

Thus, we have shown that (4.3) is always exact in the middle. Combined with Corollary 0.56 we conclude that all $d_i^{\lambda^n}$, d_i^{ρ} are trivial for $i > 0$. ☐

Remark 4.4. From the proof above, we see that in (4.3), λ^n is surjective or trivial, while ρ is injective or trivial.

4.2. **Delta extensions.**

However, the δ-extension spectral sequence for

$$\delta : \nu X/\lambda^n \to \Sigma^{1,-n}\nu X/\lambda^{m-n}$$

is more complicated, as it encodes classical Adams differentials d_2 through d_m.

Remark 4.5. For the convenience of readers to check gradings, whenever we write

$$d_n^f(x) = \lambda^k y$$

for $f : \Sigma^{m,w}\nu X \to \nu Y$, $x \in E_\infty^{s_1,t_1,w_1}(\nu X)$ and $y \in E_\infty^{s_2,t_2,w_2}(\nu Y)$, the following conditions must hold:

$$s_2 = s_1 + n,\ t_2 - s_2 = t_1 - s_1 + m, \text{ and } w_2 = w_1 + w.$$

Proposition 4.6. *Suppose in the classical Adams spectral sequence of X we have $d_r(x) = y$, where $x \in Z_{r-1}^{s,t}(X)$ and $y \in Z_\infty^{s+r,t+r-1}(X)$. Consider the map*

$$\delta : \nu X/\lambda^n \to \Sigma^{1,-n}\nu X/\lambda^{m-n}.$$

1. *If $r \geq n+1$, then we view x as an element of*
$$E_\infty^{s,t,t}(\nu X/\lambda^n) \cong Z_n^{s,t}(X),$$
and $\lambda^{r-n-1}y$ as an element of
$$E_\infty^{s+r,t+r-1,t+n}(\nu X/\lambda^{m-n}) \cong Z_{m-r+1}^{s+r,t+r-1}(X)/B_{r-n}^{s+r,t+r-1}(X).$$
We then have
$$d_r^\delta(x) = \lambda^{r-n-1}y,$$
which is trivial if $r > m$.
2. *If $r < n+1$, then we view $\lambda^{n+1-r}x$ as an element of*
$$E_\infty^{s,t,t-n-1+r}(\nu X/\lambda^n) \cong Z_{r-1}^{s,t}(X)/B_{n+2-r}^{s,t}(X),$$
and y as an element of
$$E_\infty^{s+r,t+r-1,t+r-1}(\nu X) \cong Z_\infty^{s+r,t+r-1}(X).$$
In this case, we have
$$d_r^\delta(\lambda^{n+1-r}x) = y.$$

Proof. Since $\delta = \delta_{n,m}$ is the composition of ρ and $\delta_{n,\infty}$ as the following

$$\begin{tikzcd}
\nu X/\lambda^n \arrow[r, "\delta_{n,m}"] \arrow[d, "\delta_{n,\infty}"'] & \Sigma^{1,-n}\nu X/\lambda^{m-n} \arrow[d, equal] \\
\Sigma^{1,-n}\nu X \arrow[r, "\rho"] & \Sigma^{1,-n}\nu X/\lambda^{m-n}
\end{tikzcd}$$

it suffices to prove the case when $m = \infty$ by Corollary 0.55. In the rest of the proof we will write $\delta_n = \delta_{n,\infty}$.

First, we prove by induction on n that

$$d_r^{\delta_n}(x) \equiv \lambda^{r-n-1}y \mod B_{r-1}^{s+r,t+r-1}(X) \tag{4.7}$$

when $r \geq n+1$, and

$$d_r^{\delta_n}(\lambda^{n+1-r}x) \equiv y \mod B_{r-1}^{s+r,t+r-1}(X) \tag{4.8}$$

when $r \leq n+1$. (These two expressions coincide when $r = n+1$.) For $n = 1$, the claim holds since the τ-Bockstein spectral sequence is isomorphic to the classical Adams spectral sequence. Now, assume $n \geq 2$ and the claim holds for $n-1$.

Consider the following commutative diagram.

$$\begin{tikzcd}
\Sigma^{0,-1}\nu X/\lambda^{n-1} \arrow[r, "\lambda"] \arrow[d, "\delta_{n-1}"'] & \nu X/\lambda^n \arrow[d, "\delta_n"] \\
\Sigma^{1,-n}\nu X \arrow[r, equal] & \Sigma^{1,-n}\nu X
\end{tikzcd}$$

By Corollary 0.54, if $r \leq n$, we have

$$d_r^{\delta_n}(\lambda^{n+1-r}x) = d_r^{\delta_{n-1}}(\lambda^{n-r}x) \equiv y \mod B_{r-1}^{s+r,t+r-1}(X).$$

If $r \geq n+1$, x can be viewed as an element of the E_∞-pages of $\nu X/\lambda^{n-1}$ or $\nu X/\lambda^n$. We then have

$$\lambda d_r^{\delta_n}(x) = d_r^{\delta_n}(\lambda x) = d_r^{\delta_{n-1}}(x) \equiv \lambda^{r-n} y \mod B_{r-1}^{s+r,t+r-1}(X)$$

which implies

$$d_r^{\delta_n}(x) \equiv \lambda^{r-n-1} y \mod B_{r-1}^{s+r,t+r-1}(X),$$

since $r-n+1 \leq r-1$ and hence the indeterminacy B_{r-n+1} introduced by dividing λ is contained in B_{r-1}. The induction for (4.7) and (4.8) is now complete.

We will show that $B_{r-1}^{s+r,t+r-1}(X)$ in both equations are actually equal to the sum of images of $d_0^{\delta_n}$ through $d_{r-1}^{\delta_n}$. Consider any $y' \in B_{r-1}^{s+r,t+r-1}(X)$ and assume that $d_{r'}x' = y'$ is an essential classical Adams differential for $2 \leq r' \leq r-1$. If $r \geq n+1$, using (4.7), we have

$$d_{r'}^{\delta_n}(\lambda^{r-r'}x) \equiv \lambda^{r-n-1} y' \mod B_{r'-1}^{s+r,t+r-1}(X),$$

and if $r \leq r'+1$, using (4.8), we have

$$d_{r'}^{\delta_n}(\lambda^{n+1-r'}x') \equiv y' \mod B_{r'-1}^{s+r,t+r-1}(X).$$

Notice that the extra indeterminacy here is $B_{r'-1}$ instead of B_{r-1}. By induction on r, this shows that $B_{r-1}^{s+r,t+r-1}(X)$ equals the sum of images of $d_0^{\delta_n}$ through $d_{r-1}^{\delta_n}$.

Therefore, we can omit $B_{r-1}^{s+r,t+r-1}(X)$ and simply write

$$d_r^{\delta_n}(x) = \lambda^{r-n-1} y$$

when $r \geq n+1$, and

$$d_r^{\delta_n}(\lambda^{n+1-r}x) = y$$

when $r \leq n+1$. □

Remark 4.9. The $d_r^\delta(x)$ we calculated in Proposition 4.6 may be inessential.

4.3. **Delta ESS and Adams differentials.**

Corollary 4.10. *For x, y, δ in Proposition 4.6 we always have*

$$d_r^\delta(\lambda^a x) = \lambda^{a+r-n-1} y \tag{4.11}$$

if $0 \leq a \leq n$ and $0 \leq a+r-n-1 < m-n$ (the differential is trivial if a exceeds this range).

Remark 4.12. As indicated in the proof, the right-hand side of equation (4.11) (considered as a subset of $Z_\infty^{s+r,t+r-1}(X)$) is a coset of

$$B_{r-1}^{s+r,t+r-1}(X)$$

which is the same as the value of the classical Adams differential $d_r(x) = y$. This implies that the equation (4.11) holds and is essential if and only if $d_r(x) = y$ holds and is essential. Therefore the δ-ESS encodes the same information as the classical Adams spectral sequence.

4.4. **Crossings on the E_r-page.**

Definition 4.13. Suppose $r \geq n+1$ and $d_r(x) = y$, where

$$x \in E_2^{s,t}(X), \qquad y \in E_2^{s+r,t+r-1}(X).$$

A crossing of $d_r(x) = y$ on the E_{n+1}-page refers to an essential Adams differential

$$d_{r-a-b}(x') = y',$$

where

$$x' \in E_2^{s+a,t+a}(X), \qquad y' \in E_2^{s+r-b,t+r-b-1}(X),$$

with $0 < a \leq n-1$ and $0 \leq b \leq r-n-1$. See Figure 1.

Remark 4.14. The crossing defined here is opposite to crossings in Moss's theorem.

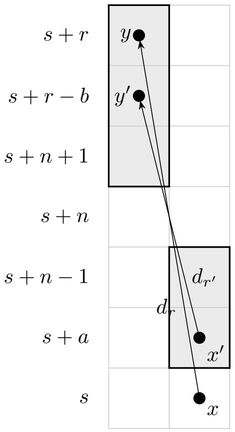

FIGURE 1. A crossing of $d_r(x) = y$ on the E_{n+1}-page

4.5. **Comparison of crossings.**

The emphasis on a crossing occurring “on the E_{n+1}-page" in Definition 4.13 may seem counter-intuitive. However, this is clarified in Proposition 4.15 and Example **??** that follow.

Proposition 4.15. *The Adams differential $d_r(x) = y$ has a crossing on the E_{n+1}-page if and only if the corresponding δ_n-extension*

$$d_r^{\delta_n}(x) = \lambda^{r-n-1} y$$

for

$$\delta_n : \nu X/\lambda^n \to \Sigma^{1,-n} \nu X$$

has a crossing.

Proof. By Propositions 0.121, 0.122 and 4.6, a crossing of $d_r^{\delta_n}(x) = \lambda^{r-n-1}y$ takes the form

$$d_{r-a-b}^{\delta_n}(\lambda^a x') = \lambda^{r-n-1-b} y',$$

where $0 < a \leq n-1$, $0 \leq b \leq r-n-1$,

$$\lambda^a x' \in E_\infty^{s+a,t+a,t}(\nu X/\lambda^n) \cong Z_{n-a}^{s+a,t+a}(X)/B_{1+a}^{s+a,t+a}(X),$$

and

$$y' \in E_\infty^{s+r-b,t+r-b-1}(\nu X).$$

By Corollary 4.10, we see that this crossing corresponds to the classical Adams differential

$$d_{r-a-b}(x') = y'.$$

□

Remark 4.16. From Definition 4.13, it immediately follows that there are no crossings of any differential on the E_2-page, as this would require $0 < a \leq n-1 = 0$. According to Proposition 4.15, this reflects the fact that δ_1-extensions have no crossings for degree reasons.

Remark 4.17. When $d_r^{\delta_n}(x) = \lambda^{r-n-1}y$ has no crossing, then $d_r^{\delta_n}(\lambda^a x) = \lambda^{a+r-n-1}y$ also has no crossing.

5. Extensions on a classical E_r-page

To state the Generalized Leibniz Rule in terms of the classical Adams spectral sequence, we need to define f-extensions not only on homotopy groups but *on the* E_r*-page* as well.

Notation 5.1. For a map between classical spectra $f : X \to Y$, consider the associated synthetic map from Notation 0.126

$$\hat{f} : \Sigma^{0,e(f)}\nu X \to \nu Y.$$

For any $2 \leq r \leq \infty$, we denote the following mod λ^{r-1} reduction maps by $\hat{f}_{r-1}$:

$$\hat{f}_{r-1} : \Sigma^{0,e(f)}\nu X/\lambda^{r-1} \to \nu Y/\lambda^{r-1}.$$

The E_0-page of the $\hat{f}_{r-1}$-ESS is isomorphic to

$$\begin{aligned} {}^{\hat{f}_{r-1}}E_0^{s,t,t-k+e(f)} &\cong E_\infty^{s,t,t-k}(\nu X/\lambda^{r-1}) \oplus E_\infty^{s,t,t-k+e(f)}(\nu Y/\lambda^{r-1}) \\ &\cong \big(Z_{r-1-k}^{s,t}(X)/B_{1+k}^{s,t}(X)\big) \oplus \big(Z_{r-1-k+e(f)}^{s,t}(Y)/B_{1+k-e(f)}^{s,t}(Y)\big). \end{aligned}$$

A nontrivial $d_n^{\hat{f}_{r-1}}$ differential can be interpreted as a map from the subgroup

$$ {}^{\hat{f}_{r-1}}Z_{n-1}^{s,t,t-k}(X) \subset Z_{r-1-k}^{s,t}(X)/B_{1+k}^{s,t}(X) \tag{5.2}$$

to the quotient group

$$\big(Z_{r-1-k-n+e(f)}^{s+n,t+n}(Y)/B_{1+k+n-e(f)}^{s+n,t+n}(Y)\big)/\, {}^{\hat{f}_{r-1}}B_{n-1}^{s+n,t+n,t-k}(Y). \tag{5.3}$$

The differential $d_n^{\hat{f}_{r-1}}$ is trivial for degree reasons when

$$n < \mathrm{AF}(f) \text{ or } n > r-2-k+e(f).$$

Definition 5.4. Let $x \in Z_{r-1}^{s,t}(X)$ and $y \in Z_{r-1-n+e(f)}^{s+n,t+n}(Y)$ for some

$$e(f) \leq n \leq r-2+e(f).$$

We say that there is an (f, E_r)-extension from x to y, denoted by

$$d_n^{f,E_r}(x) = y \tag{5.5}$$

if there exists a synthetic $\hat{f}_{r-1}$-extension

$$d_n^{\hat{f}_{r-1}}(x) = \lambda^{n-e(f)}y. \tag{5.6}$$

where x is viewed as an element of the subgroup (5.2) with $k = 0$, and $\lambda^{n-e(f)}y$ is viewed as an element of the quotient group (5.3) with $k = 0$.

We say that this (f, E_r)-extension in (5.5) is *essential* if the corresponding synthetic $\hat{f}_{r-1}$-extension in (5.6) is an essential differential in the $\hat{f}_{r-1}$-ESS.

For $r = \infty$, we similar define an (f, E_∞)-extension using the corresponding synthetic $\hat{f}$-extension.

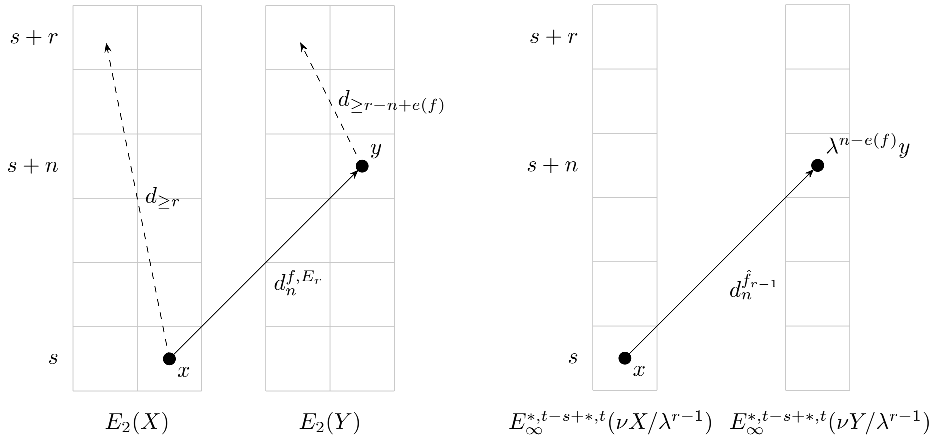

FIGURE 2. (f, E_r)-extension

Remark 5.7. Consider an (f, E_r)-extension $d_n^{f,E_r}(x) = y$ in (5.5). The element y should be interpreted as a coset with indeterminacy given by $B_{1+n-e(f)}^{s+n,t+n}(Y)$, plus the sum of images of $d_{<n}^{\hat{f}_{r-1}}$ differentials. This (f, E_r)-extension is *essential* if this coset does not contain 0.

Definition 5.8. Let $x \in Z_{r-1}^{s,t}(X)$ and $y \in Z_{r-1-n+e(f)}^{s+n,t+n}(Y)$ for some

$$e(f) \leq n \leq r - 2 + e(f).$$

We say that there is an (f, E_r)-extension of level l from x to y, denoted by

$$d_n^{f,E_r,l}(x) = y \tag{5.9}$$

if there exists a synthetic $\hat{f}_{r-1}$-extension

$$d_n^{\hat{f}_{r-1}}(\lambda^l x) = \lambda^{n-e(f)+l}y. \tag{5.10}$$

where $\lambda^l x$ is viewed as an element of the subgroup (5.2) with certain k, and $\lambda^{n-e(f)+l}y$ is viewed as an element of the quotient group (5.3).

We say that this (f, E_r)-extension of level l in (5.9) is *essential* if the corresponding synthetic $\hat{f}_{r-1}$-extension in (5.10) is an essential differential in the $\hat{f}_{r-1}$-ESS.

For $r = \infty$, we similarly define an (f, E_∞)-extension of level l using the corresponding synthetic $\hat{f}$-extension.

For $l < l'$, an (f, E_r)-extension of level l implies an extension of the same type of level l', but the latter might be inessential.

Proposition 5.11. *Suppose we have an (f, E_r)-extension of level l',*

$$d_n^{f,E_r,l'}(x) = y.$$

Then either

(1) there is an essential (f, E_r)-extension of level l, $d_m^{f,E_r,l}(x) = z$, for some $m < n$, or

(2) we have $d_n^{f,E_r,l}(x) = y + z$ for some z lying in the indeterminacy of $d_n^{f,E_r,l'}$.

Definition 5.12. A crossing of the (f, E_r)-extension $d_n^{f,E_r}(x) = y$ in (5.5) is defined as an essential (f, E_{r-a})-extension from some $x' \in Z_{r-1-a}^{s+a,t+a}(X)$ to

$$y' \in Z_{r-1-n+b+e(f)}^{s+n-b,t+n-b}(Y) \backslash B_{1+n-b-e(f)}^{s+n-b,t+n-b}(Y)$$

(where $\backslash$ denotes the difference of sets and it means that y' should survive to the classical Adams $E_{r-n+b+e(f)}$ page while it should not be hit by an Adams differential of length at most $1+n-b-e(f)$) for $0 < a \leq r-2$ and $0 \leq b \leq n-a-e(f)$. See Figure 3.

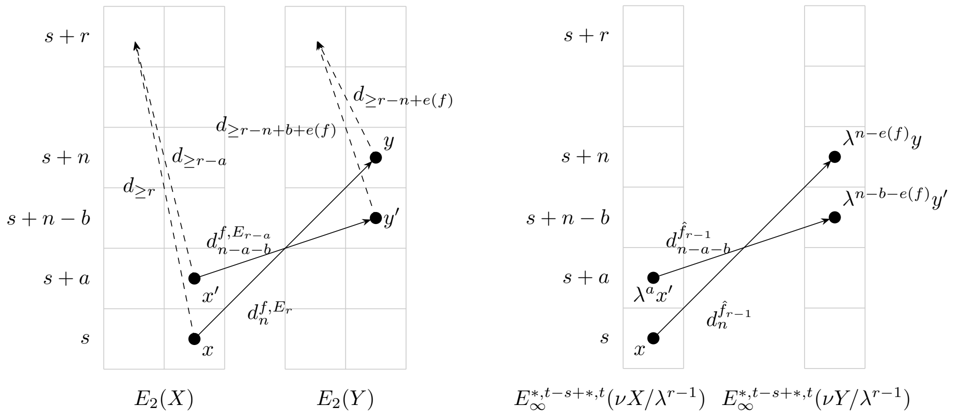

FIGURE 3. A crossing of an (f, E_r)-extension

Proposition 5.13. *An (f, E_r)-extension $d_n^{f,E_r}(x) = y$ (5.5) has a crossing if and only if the synthetic $\hat{f}_{r-1}$-extension $d_n^{\hat{f}_{r-1}}(x) = \lambda^{n-e(f)}y$ (5.6) has a crossing.*

Proof. By definition a crossing of the synthetic extension (5.6) has form

$$d_{n-a-b}^{\hat{f}_r}(\lambda^a x') = \lambda^{n-b-e(f)}y' \tag{5.14}$$

for some $x' \in Z_{r-1-a}^{s+a,t+a}(X)$ and $y' \in Z_{r-1-n+b+e(f)}^{s+n-b,t+n-b}(Y)$. Consider the following commutative diagram

$$\begin{tikzcd}
\Sigma^{0,e(f)}\nu X/\lambda^{r-1-a} \arrow[r, "\hat{f}_{r-1-a}"] \arrow[d, "\lambda^a"'] & \nu Y/\lambda^{r-1-a} \arrow[d, "\lambda^a"] \\
\Sigma^{0,e(f)+a}\nu X/\lambda^{r-1} \arrow[r, "\hat{f}_{r-1}"] & \Sigma^{0,a}\nu Y/\lambda^{r-1}
\end{tikzcd}$$

We see that (5.14) lifts to

$$d_{n-a-b}^{\hat{f}_{r-1-a}}(x') = \lambda^{n-a-b-e(f)}y'' \tag{5.15}$$

for some $y'' \in Z_{r-1-n+b+e(f)}^{s+n-b,t+n-b}(Y)$ such that

$$y'' \equiv y' \mod B_{1+n-b-e(f)}^{s+n-b,t+n-b}(Y)$$

where the indeterminacy $B^{s+n-b,t+n-b}_{1+n-b-e(f)}(Y)$ is introduced by dividing λ^a. Therefore, the differential (5.14) is essential if and only if the differential (5.15) is essential and $y'' \notin B^{s+n-b,t+n-b}_{1+n-b-e(f)}(Y)$. This completes the proof. ☐

The above Definition 5.12 and Proposition 5.13 extend to the case $r = \infty$ as follows.

Definition 5.16. A crossing of the (f, E_∞)-extension $d_n^{f,E_\infty}(x) = y$ is defined as an essential (f, E_∞)-extension from some $x' \in Z^{s+a,t+a}_\infty(X)$ to

$$y' \in Z^{s+n-b,t+n-b}_\infty(Y)\backslash B^{s+n-b,t+n-b}_{1+n-b-e(f)}(Y)$$

(where $\backslash$ denotes the difference of sets, meaning y' survives to the classical Adams E_∞-page while it is not hit by any Adams differential) for $a > 0$ and $0 \leq b \leq n - a - e(f)$.

Proposition 5.17. *An (f, E_∞)-extension $d_n^{f,E_\infty}(x) = y$ has a crossing if and only if the synthetic $\hat{f}$-extension $d_n^{\hat{f}}(x) = \lambda^{n-e(f)}y$ has a crossing.*

Similarly, the crossing notion extends to extensions of level l.

Definition 5.18. A crossing of the (f, E_r)-extension of level l $d_n^{f,E_r,l}(x) = y$ in (5.9) is defined as an essential (f, E_{r-a})-extension of level l from some $x' \in Z^{s+a,t+a}_{r-1-a}(X)$ to

$$y' \in Z^{s+n-b,t+n-b}_{r-1-n+b+e(f)}(Y)\backslash B^{s+n-b,t+n-b}_{1+n-b-e(f)}(Y)$$

for $0 < a \leq r - 2$ and $0 \leq b \leq n - a - e(f)$.

For $r = \infty$, we similarly define a crossing of an (f, E_∞)-extension of level l as an essential (f, E_∞)-extension of level l from some $x' \in Z^{s+a,t+a}_\infty(X)$ to $y' \in Z^{s+n-b,t+n-b}_\infty(Y)\backslash B^{s+n-b,t+n-b}_{1+n-b-e(f)}(Y)$ for $a > 0$ and $0 \leq b \leq n - a - e(f)$.

Proposition 5.19. *An (f, E_r)-extension of level l $d_n^{f,E_r,l}(x) = y$ (5.9) has a crossing if and only if the corresponding synthetic $\hat{f}_{r-1}$-extension $d_n^{\hat{f}_{r-1}}(\lambda^l x) = \lambda^{n-e(f)+l}y$ (5.10) has a crossing.*

Proposition 5.20. *If an (f, E_r)-extension of level l, $d_n^{f,E_r,l}(x) = y$, has a crossing, then there is an essential (f, E_r)-extension of level 0, $d_m^{f,E_r,0}(x') = z'$, such that the implied extension at level $a + l$ satisfies the degree conditions of a crossing.*

Proof. If there is a crossing, we apply the alternatives of Proposition 5.11. ☐

Proposition 5.21. *If an (f, E_r)-extension of level l, $d_n^{f,E_r,l}(x) = y$, has no crossing, then the induced extension at any level $l' > l$ also has no crossing.*

Proposition 5.22. *Consider $f : X \to Y$, $x \in Z^{s,t}_\infty(X)$ and $y \in Z^{s+n,t+n}_\infty(Y)$. Then*

$$d_n^{f,E_\infty}(x) = y$$

implies that

$$d_n^f(x + B^{s,t}_\infty(X)) = y + B^{s,t}_\infty(Y).$$

(The implied differential could be inessential.)

Proof. This follows directly from inverting λ, which induces a map of spectral sequences from the $\hat{f}$-ESS to the f-ESS. The induced map on the E_0-pages are of the following form:

$$\begin{tikzcd}
Z_\infty^{s,t}(X)/B_{1+t-w}^{s,t}(X) \arrow[r, "d_0^{\hat{f}}"] \arrow[d, "\lambda^{-1}"'] & Z_\infty^{s,t}(Y)/B_{1+t-w-e(f)}^{s,t}(Y) \arrow[d, "\lambda^{-1}"] \\
E_\infty^{s,t}(X) \arrow[r, "d_0^f"] & E_\infty^{s,t}(Y)
\end{tikzcd}$$

□

Remark 5.23. Since the synthetic Adams E_∞-page contains more information than the classical Adams E_∞-page, the (f, E_∞)-extensions similarly provide more information compared to the classical f-extensions.

Proposition 5.24. *Consider $f : X \to Y$, $x \in Z_\infty^{s,t}(X)$ and $y \in Z_\infty^{s+n,t+n}(Y)$. Then*

$$d_n^{f,E_\infty,l}(x) = y$$

implies that

$$d_n^f(x + B_\infty^{s,t}(X)) = y + B_\infty^{s,t}(Y).$$

Proof. The same as Proposition 5.22, since we invert λ. □

6. The Generalized Leibniz Rule and Generalized Mahowald Trick

Now, we introduce the theorem of the Generalized Leibniz Rule, a valuable tool for computing new Adams differentials.

Theorem 6.1 (Generalized Leibniz Rule)**.** *Let $f : X \to Y$ be a map between two classical spectra. Suppose that $2 \le n \le r$, $e(f) \le m \le n - 2 + e(f)$, $l \ge e(f)$, and we have*

$$x \in Z_{r-1}^{s,t}(X), \quad y \in Z_{r-1-m+e(f)}^{s+m,t+m}(Y)$$
$$x_\infty \in Z_\infty^{s+r,t+r-1}(X), \quad y_\infty \in Z_\infty^{s+r+l,t+r+l-1}(Y)$$

and the following conditions hold:

1. *$d_r(x) = x_\infty$,*
2. *$d_m^{f,E_n}(x) = y$,*
3. *$d_l^{f,E_\infty}(x_\infty) = y_\infty$,*
4. *the differential in (1) has no crossing on the E_n-page or (2) has no crossing.*
5. *the differential in (3) has no crossing.*

Then we have an Adams differential

$$d_{r+l-m}(y) = y_\infty.$$

Proof. Consider the commutative diagram of synthetic spectra

$$\begin{tikzcd}
\Sigma^{0,e(f)}\nu X/\lambda^{n-1} \arrow[r, "\hat{f}_{n-1}"] \arrow[d, "\delta_X"'] & \nu Y/\lambda^{n-1} \arrow[d, "\delta_Y"] \\
\Sigma^{1,-n+1+e(f)}\nu X \arrow[r, "\hat{f}"] & \Sigma^{1,-n+1}\nu Y
\end{tikzcd}$$

By condition (1) and Proposition 4.6, we have

$$d_r^{\delta_X}(x) = \lambda^{r-n} x_\infty. \tag{6.2}$$

By conditions (2) and (3), and Definition 5.4, we have

$$d_m^{\hat{f}_{n-1}}(x) = \lambda^{m-e(f)}y \tag{6.3}$$

and

$$d_l^{\hat{f}}(x_\infty) = \lambda^{l-e(f)}y_\infty,$$

which implies

$$d_l^{\hat{f}}(\lambda^{r-n}x_\infty) = \lambda^{r+l-n-e(f)}y_\infty. \tag{6.4}$$

Applying Proposition 4.15 and Proposition 5.13 to conditions (4) and (5), we know that the differential in (6.2) or (6.3) has no crossing, and that the differential in (6.4) has no crossing.

Therefore, we can apply Corollary 0.53 and obtain

$$d_{r+l-m}^{\delta_Y}(\lambda^{m-e(f)}y) = \lambda^{r+l-n-e(f)}y_\infty.$$

By Remark 4.12, this is equivalent to

$$d_{r+l-m}(y) = y_\infty.$$

□

Remark 6.5. We can further generalize Theorem 6.1 by using the conditions in Theorem 0.52 rather than those Corollary 0.53. We leave this generalization to the reader.

Theorem 6.6 (Generalized Leibniz Rule, level k)**.** *Let $f : X \to Y$ be a map between two classical spectra. Suppose that $2 \leq n \leq r$, $e(f) \leq m \leq n-2+e(f)$, $l \geq e(f)$, $k \geq 0$, and we have*

$$x \in Z_{r-1}^{s,t}(X), \qquad y \in Z_{r-1-m+e(f)}^{s+m,t+m}(Y)$$
$$x_\infty \in Z_\infty^{s+r,t+r-1}(X), \qquad y_\infty \in Z_\infty^{s+r+l,t+r+l-1}(Y)$$

and the following conditions hold:

1. *$d_r(x) = x_\infty$,*
2. *$d_m^{f,E_n,k}(x) = y$,*
3. *$d_l^{f,E_\infty,k}(x_\infty) = y_\infty$,*
4. *the differential in (1) has no crossing on the E_n-page or (2) has no crossing.*
5. *the differential in (3) has no crossing.*

Then we have an Adams differential

$$d_{r+l-m}(y) = y_\infty.$$

Proof. Consider the commutative diagram of synthetic spectra

$$\begin{tikzcd}
\Sigma^{0,e(f)}\nu X/\lambda^{n-1} \arrow[r, "\hat{f}_{n-1}"] \arrow[d, "\delta_X"'] & \nu Y/\lambda^{n-1} \arrow[d, "\delta_Y"] \\
\Sigma^{1,-n+1+e(f)}\nu X \arrow[r, "\hat{f}"] & \Sigma^{1,-n+1}\nu Y
\end{tikzcd}$$

By condition (1) and Proposition 4.6, we have

$$d_r^{\delta_X}(x) = \lambda^{r-n}x_\infty. \tag{6.7}$$

By conditions (2) and (3), and Definition 5.8, we have

$$d_m^{\hat{f}_{n-1}}(\lambda^k x) = \lambda^{m-e(f)+k}y \tag{6.8}$$

and

$$d_l^{\hat{f}}(\lambda^k x_\infty) = \lambda^{l-e(f)+k} y_\infty,$$

which implies

$$d_l^{\hat{f}}(\lambda^{r-n+k} x_\infty) = \lambda^{r+l-n-e(f)+k} y_\infty. \tag{6.9}$$

Applying Proposition 4.15 and Proposition 5.19 to conditions (4) and (5), we know that the differential in (6.7) or (6.8) has no crossing, and that the differential in (6.9) has no crossing.

Therefore, we can apply Corollary 0.53 and obtain

$$d_{r+l-m}^{\delta_Y}(\lambda^{m-e(f)+k} y) = \lambda^{r+l-n-e(f)+k} y_\infty.$$

By Remark 4.12, this is equivalent to

$$d_{r+l-m}(y) = y_\infty.$$

□

Theorem 6.10 (Generalized Leibniz Rule, first direction)**.** *Let $f : X \to Y$ be a map between two classical spectra. Suppose that $2 \leq n \leq r$, $e(f) \leq m \leq n-2+e(f)$, $l \geq e(f)$, $k \geq 0$, and we have*

$$x \in Z_{r-1}^{s,t}(X), \qquad y \in Z_{r-1-m+e(f)}^{s+m,t+m}(Y)$$
$$x_\infty \in Z_\infty^{s+r,t+r-1}(X), \qquad y_\infty \in Z_\infty^{s+r+l,t+r+l-1}(Y)$$

and the following conditions hold:

(1) $d_r(x) = x_\infty$,
(2) $d_m^{f,E_n,k}(x) = y$,
(3) $d_{r+l-m}(y) = y_\infty$,
(4) the differential in (1) has no crossing on the E_n-page or (2) has no crossing,
(5) the differential in (3) has no crossing.

Then we have an (f, E_∞)-extension

$$d_l^{f,E_\infty,r-n+k}(x_\infty) = y_\infty.$$

Proof. Consider the commutative diagram of synthetic spectra

$$\begin{tikzcd}
\Sigma^{0,e(f)}\nu X/\lambda^{n-1} \arrow[r, "\hat{f}_{n-1}"] \arrow[d, "\delta_X"'] & \nu Y/\lambda^{n-1} \arrow[d, "\delta_Y"] \\
\Sigma^{1,-n+1+e(f)}\nu X \arrow[r, "\hat{f}"] & \Sigma^{1,-n+1}\nu Y
\end{tikzcd}$$

By condition (1) and Proposition 4.6, we have

$$d_r^{\delta_X}(\lambda^k x) = \lambda^{r-n+k} x_\infty. \tag{6.11}$$

By condition (2) and Definition 5.8, we have

$$d_m^{\hat{f}_{n-1}}(\lambda^k x) = \lambda^{m-e(f)+k} y. \tag{6.12}$$

By condition (3) and Remark 4.12, we have

$$d_{r+l-m}^{\delta_Y}(\lambda^{m-e(f)+k} y) = \lambda^{r+l-n-e(f)+k} y_\infty. \tag{6.13}$$

Applying Proposition 4.15 and Proposition 5.19 to conditions (4) and (5), we know that the differential in (6.11) or (6.12) has no crossing, and that the differential in (6.13) has no crossing.

Therefore, we can apply Corollary 0.53 to the transposed commutative diagram (with δ_X as the first map, $\hat{f}_{n-1}$ as the second map, and δ_Y as the known target extension) and obtain

$$d_l^{\hat{f}}(\lambda^{r-n+k}x_\infty) = \lambda^{r+l-n-e(f)+k}y_\infty.$$

By definition, this is equivalent to

$$d_l^{f,E_\infty,r-n+k}(x_\infty) = y_\infty.$$

□

We emphasize that the no-crossing conditions in Theorem 6.1, the Generalized Leibniz Rule, are crucial. Without the no-crossing condition, the conclusion could be false.

Remark 6.14. A version of the Generalized Leibniz Rule without the no-crossing conditions is presented in the synthetic setting in Chua's work [8, Theorem 12.9]. However, there is no doubt that this version is incorrect. For further details, see Remark 6.16.

Remark 6.15. The essential (f, E_∞)-extension:

$$d_1^{f,E_\infty}(d_0) = h_0d_0,$$

is a crossing for both the (f, E_∞)-extension,

$$d_2^{f,E_\infty}(h_0h_3^2) = 0,$$

and the inessential (f, E_∞)-extension:

$$d_2^{f,E_\infty}(h_0h_3^2) = h_0d_0.$$

This indicates that if we were to disregard the no-crossing condition (5), and apply the Generalized Leibniz Rule to these two cases of the (f, E_∞)-extensions, we would arrive at two conflicting classical statements:

$$d_3(h_0h_4) = 0,\ d_3(h_0h_4) = h_0d_0.$$

Remark 6.16. In Chua's work [8, Theorem 12.9], it is stated that for a synthetic map $\alpha : X \to Y$, and an element $x \in \pi_{*,*}X/\lambda$, there exists a differential from a maximal α-extension of x to a maximal α-extension of $d_r(x)$. According to [8, Definition 12.5], a maximal α-extension of x' is defined as $\alpha[x']$, where $[x']$ represents a lift of x' to the $E_{r'}$-page, chosen such that $\alpha[x']$ is the most λ-divisible among all such lifts.

In the context of the counter-example above, let $r = 2, r' = \infty$,

$$\alpha = [h_0] : S^{0,1} \to S^{0,0},$$

and consider $x = h_4$ in Ext, with $x' = d_2(x) = h_0h_3^2$.

The (f, E_∞)-extension:

$$d_2^{f,E_\infty}(h_0h_3^2) = 0$$

is equivalent to the existence of a synthetic homotopy class $[h_0h_3^2]$ in $\pi_{14,17}S^{0,0}$, such that

$$[h_0h_3^2] \cdot [h_0] = 0 \text{ in } \pi_{14,18}S^{0,0}.$$

Similarly, the inessential (f, E_∞)-extension:

$$d_2^{f,E_\infty}(h_0h_3^2) = h_0d_0$$

implies the existence of another synthetic homotopy class $[h_0h_3^2]$ in $\pi_{14,17}S^{0,0}$, such that

$$[h_0h_3^2] \cdot [h_0] = [h_0d_0] \text{ in } \pi_{14,18}S^{0,0}.$$

Between these two lifts of $h_0h_3^2$, the first $[h_0h_3^2]$ is clearly the maximal $[h_0]$-extension according to Chua's definition [8, Definition 12.5]. Consequently, the incorrect version of the Generalized Leibniz Rule in [8, Theorem 12.9], would lead to an incorrect conclusion:

$$d_3(h_0h_4) = 0.$$

Next, we discuss the Generalized Mahowald Trick.

In order to apply the Generalized Leibniz Rule, we need to provide a method for computing extensions on specific Adams E_k-pages. This is provided by Theorem 6.18 (the Generalized Mahowald Trick). The crux of the proof of the Generalized Mahowald Trick lies in the following lemma.

Lemma 6.17 (May [17])**.** *Let $X \to Y \to Z \to \Sigma X$ and $X' \to Y' \to Z' \to \Sigma X'$ be distinguished triangles of (synthetic) spectra. By smashing these distinguished triangles together, we obtain the following commutative diagram of cofiber sequences:*

$$\begin{tikzcd}
X \wedge X' \arrow[r] \arrow[d] & Y \wedge X' \arrow[r] \arrow[d] & Z \wedge X' \arrow[d] \\
X \wedge Y' \arrow[r] \arrow[d] & Y \wedge Y' \arrow[r] \arrow[d] & Z \wedge Y' \arrow[d] \\
X \wedge Z' \arrow[r] & Y \wedge Z' \arrow[r] & Z \wedge Z'
\end{tikzcd}$$

If $a \in \pi_n(X \wedge Z')$ and $b \in \pi_n(Y \wedge Y')$ map to the same element in $\pi_n(Y \wedge Z')$, then there exists $c \in \pi_n(Z \wedge X')$ such that

1. *(1) b and c map to the same element in $\pi_n(Z \wedge Y')$, and*
2. *(2) a and c map to the same element via boundary maps in $\pi_{n-1}(X \wedge X')$.*

Proof. The proof follows directly from [17, Lemma 4.6].

In fact, consider V in Axiom TC3 of [17, Section 4] and the corresponding commutative diagram. By [17, Lemma 4.6] we know that V is the pull back of the following diagram.

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & Y \wedge Y' \arrow[d] \\
X \wedge Z' \arrow[r] & Y \wedge Z'
\end{tikzcd}$$

where the square is a pullback. Since a and b map to the same element in $\pi_n(Y \wedge Z')$, we know that we can find $v \in V$ such that v maps to a in $\pi_n(X \wedge Z')$ and b in $\pi_n(Y \wedge Y')$. Then we let $c = j_3(v) \in \pi_n(Z \wedge X')$, where $j_3 : V \to Z \wedge X'$ is the map in the commutative diagram in Axiom TC3 of [17, Section 4]. This lemma follows from the commutativity of the diagram. □

Theorem 6.18 (Generalized Mahowald Trick)**.** *Consider a distinguished triangle of spectra*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

with $e(f)+e(g)+e(h)=1$. *Suppose that* $r=n+m+l$, $n_1=n-e(f)\geq 1$, $m_1=m-e(g)\geq 0$, $l_1=l-e(h)\geq 0$, *and*

$$\begin{array}{ll} x\in Z_{n_1}^{s+l,t+l-1}(X), & y\in Z_{m_1+1}^{s+n+l,t+n+l-1}(Y), \\ \bar{x}\in Z_{r-1}^{s,t}(Z), & \bar{y}\in Z_{\infty}^{s+r,t+r-1}(Z), \end{array}$$

such that

1. $d_l^{h,E_{r'}}\bar{x}=x$, *where* $r'=r-m_1=n_1+l_1+1$,
2. $d_r\bar{x}=\bar{y}$,
3. *the* $(h,E_{r'})$*-extension in (1) has no crossing, or the Adams differential (2) has no crossing on the* $E_{r'}$*-page,*
4. $d_m^{g,E_{m_1+2}}y=\bar{y}$,
5. *for* $0\leq i\leq n-1$*, the Adams* E_∞*-page of* Y *vanishes at the positions that are potential targets of* (f,E_r)*-extensions of degree* i *from* x: $E_\infty^{s+l+i,t+l+i-1}(Y)=0$.

Then we have $x\in Z_{n+m+e(h)}^{s+l,t+l-1}(X)$ *and*

$$d_n^{f,E_{n+m+1+e(h)}}x\equiv y \mod B_{r'}^{s+n+l,t+n+l-1}(Y).$$

(See Figure 4.)

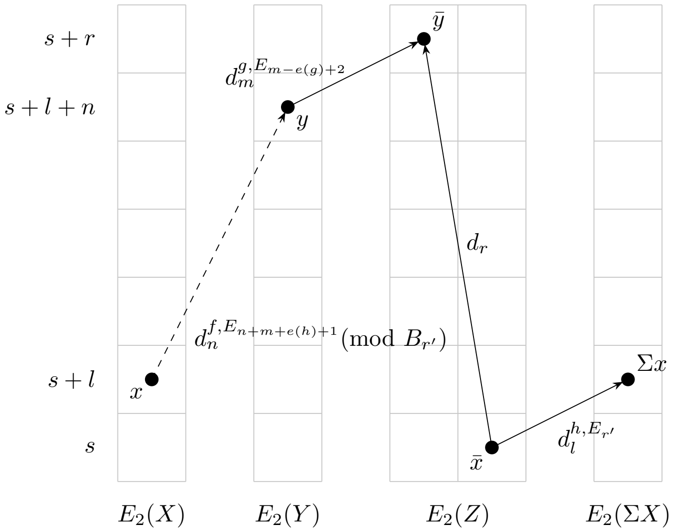

FIGURE 4. A demonstration of Theorem 6.18

Proof. Consider the following two distinguished triangles of synthetic spectra

$$\nu X\xrightarrow{\hat{f}}\Sigma^{0,-e(f)}\nu Y\xrightarrow{\hat{g}}\Sigma^{0,-e(f)-e(g)}\nu Z\xrightarrow{\hat{h}}\Sigma^{0,-1}\nu\Sigma X=\Sigma^{1,0}\nu X$$

$$S^{0,0}/\lambda^r\xrightarrow{\rho}S^{0,0}/\lambda^{r'-1}\xrightarrow{\delta_{r'-1}}S^{1,-r'+1}/\lambda^{m_1+1}\xrightarrow{\lambda^{r'-1}}S^{1,0}/\lambda^r$$

and their smash product. See Figure 5.

By condition (1), we have

$$d_l^{\hat{h}_{r'-1}}(\bar{x})=\lambda^{l_1}x \tag{6.19}$$

$$\begin{tikzcd}
\bigwedge & \nu X \arrow[r, "\hat{f}"] & \Sigma^{0,-e(f)}\nu Y \arrow[r, "\hat{g}"] & \Sigma^{0,-e(f)-e(g)}\nu Z \arrow[r, "\hat{h}"] & \Sigma^{1,0}\nu X \\
S^{0,0}/\lambda^{r} \arrow[d, "\rho"'] & & & & {[\lambda^{l_1}x]} \arrow[d] \\
S^{0,0}/\lambda^{r'-1} \arrow[d, "\delta_{r'-1}"'] & & {[\bar{x}]} \arrow[r] & {[\lambda^{l_1}x]} \arrow[d] & {} \\
S^{1,-r'+1}/\lambda^{m_1+1} \arrow[d, "\lambda^{r'-1}"'] & {[y]} \arrow[r] & {[\lambda^{m_1}\bar{y}]} \arrow[d] & {} & \\
S^{1,0}/\lambda^{r} & {[\lambda^{l_1}x]} \arrow[r] & {[\lambda^{r'-1}y]} & &
\end{tikzcd}$$

FIGURE 5. Elements in the homotopy groups of the smash products

where $\hat{h}_{r'-1}$ is the map $\Sigma^{0,e(h)}\nu Z/\lambda^{r'-1} \to \nu X/\lambda^{r'-1}$ induced by $\hat{h}$. By condition (2), we have

$$d_r^{\delta_{r'-1}}(\bar{x}) = \lambda^{m_1}\bar{y}. \tag{6.20}$$

Applying Proposition 4.15 and Proposition 5.13 to condition (3), we know that the differential in (6.19) or the differential in (6.20) has no crossing. Hence, there exists

$$[\bar{x}] \in \{\bar{x}\} \subset \pi_{t-s,t}(\nu Z/\lambda^{r'-1})$$

such that

$$\hat{h}_{r'-1}([\bar{x}]) \in \{\lambda^{l_1}x\} \subset \pi_{t-s-1,t-1+e(h)}(\nu X/\lambda^{r'-1})$$

and

$$\delta_{r'-1}([\bar{x}]) \in \{\lambda^{m_1}\bar{y}\} \subset \pi_{t-s-1,t+r'-1}(\nu Z/\lambda^{m_1+1}).$$

By condition (4), we have

$$d_m^{\hat{g}_{m_1+1}}(y) = \lambda^{m_1}\bar{y}.$$

This implies that there exists

$$[y] \in \{y\} \subset \pi_{t-s-1,t+n+l-1}(\nu Y/\lambda^{m_1+1})$$

such that

$$\hat{g}_{m_1+1}([y]) \in \{\lambda^{m_1}\bar{y}\} \subset \pi_{t-s-1,t+r'-1}(\nu Z/\lambda^{m_1+1}),$$

where $\hat{g}_{m_1+1}$ is the map $\Sigma^{0,e(g)}\nu Y/\lambda^{m_1+1} \to \nu Z/\lambda^{m_1+1}$ induced by $\hat{g}$.

Due to degree reasons (Proposition 0.122), $\lambda^{m_1}\bar{y}$ detects a unique element in homotopy,

$$[\lambda^{m_1}\bar{y}] \in \pi_{t-s-1,t+r'-1}(\nu Z)/\lambda^{m_1+1}.$$

Therefore, we have

$$\delta_{r'-1}([\bar{x}]) = \hat{g}_{m_1+1}([y]) = [\lambda^{m_1}\bar{y}],$$

By Lemma 6.17, there exists

$$[\lambda^{l_1}x] \in \pi_{t-s-1,t-1+e(h)}(\nu X/\lambda^{r}),$$

such that

$$\rho([\lambda^{l_1}x]) = \hat{h}_{r'-1}([\bar{x}]), \tag{6.21}$$

$$\hat{f}_r([\lambda^{l_1}x]) = \lambda^{r'-1}[y], \tag{6.22}$$

where ρ is the map $\nu X/\lambda^r \to \nu X/\lambda^{r'-1}$, and $\hat{f}_r$ is the map $\Sigma^{0,e(f)}\nu X/\lambda^r \to \nu Y/\lambda^r$ induced by $\hat{f}$.

The equality in (6.21) indicates that $[\lambda^{l_1}x]$ can be lifted to the E_∞-page of $\nu X/\lambda^r$, so we have

$$x \in Z^{s+l,t+l-1}_{n+m+e(h)}(X).$$

The equation in (6.22) indicates that

$$d^{\hat{f}_r}_n(\lambda^{l_1}x) = \lambda^{r'-1}y.$$

Therefore, by dividing λ^{l_1}, we have

$$d_n^{\hat{f}_{n+m+e(h)}}(x) \equiv \lambda^{n_1}y \mod B^{s+n+l,t+n+l-1}_{r'}(Y)$$

for x in the E_∞ page of $\nu X/\lambda^{r-l_1} = \nu X/\lambda^{n+m+e(h)}$. By definition, this is equivalent to

$$d_n^{f,E_{n+m+1+e(h)}}x \equiv y \mod B^{s+n+l,t+n+l-1}_{r'}(Y).$$

□

Remark 6.23. There is also a version of the Generalized Mahowald Trick where the assumptions involve (g, E_{m_1+2})-extensions and $(h, E_{r'})$-extensions of level k. We leave this generalization to the reader.

We refer to Theorem 6.18 as the Generalized Mahowald Trick, as this approach was first utilized by Mahowald and his collaborators in various works (see, for example, [16]), particularly in the case where $m_1 = l_1 = 0$. The synthetic setting advances this method by allowing for the consideration of cases where $m_1 > 0, l_1 > 0$ as well.

Remark 6.24. Several versions of classical generalizations of Mahowald's original trick appear in the literature and are often referred to as geometric boundary theorems. Notable examples include the works of Behrens [2], and Ma [15].

Remark 6.25. A synthetic generalization of the Mahowald Trick, again lacking any no-crossing conditions, is also presented in Chua's work [8, Theorem 12.11]. This version is also incorrect for similar reasons.

The outcome of the Generalized Mahowald Trick, as stated in Theorem 6.18, is an f-extension on a specific Adams page. In practice, the source of an (f, E_r)-extension may survive to later Adams pages, prompting interest in the $(f, E_{>r})$-extensions. The following Propositions 6.26 and 6.29 describe the relationships between extensions across different pages.

Proposition 6.26. *Suppose that we have an (f, E_r)-extension $d_n^{f,E_r}(x) = y$, where $x \in Z^{s,t}_{r-1}(X)$ and $y \in Z^{s+n,t+n}_{r-1-n+e(f)}(Y)$. Then for all $2 \le r' < r$, we also have*

$$d_n^{f,E_{r'}}(x) = y. \tag{6.27}$$

Furthermore, if $d_n^{f,E_r}(x) = y$ is essential and $n \le r' - 2 + e(f)$, then (6.27) is inessential if and only if there exists some $0 < a' \le n - e(f)$ and an element

$$x' \in Z^{s+a',t+a'}_{r'-1-a'}(X) \backslash Z^{s+a',t+a'}_{r-1-a'}(X)$$

such that

$$d^{\hat{f}_{r'-1}}_{n-a'-b}(\lambda^{a'}x') = \lambda^{n-b-e(f)}y'$$

for some $b \ge 0$.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
\Sigma^{0,e(f)}\nu X/\lambda^{r-1} \arrow[r, "\hat{f}_{r-1}"] \arrow[d, "\rho_X"'] & \nu Y/\lambda^{r-1} \arrow[d, "\rho_Y"] \\
\Sigma^{0,e(f)}\nu X/\lambda^{r'-1} \arrow[r, "\hat{f}_{r'-1}"] & \nu Y/\lambda^{r'-1}
\end{tikzcd}$$

By Corollary 0.57, (ρ_X, ρ_Y) induces a map from the $\hat{f}_{r-1}$-ESS to the $\hat{f}_{r'-1}$-ESS. Therefore, by naturality,

$$d_n^{\hat{f}_{r-1}}(x) = \lambda^{n-e(f)}y \text{ implies } d_n^{\hat{f}_{r'-1}}(x) = \lambda^{n-e(f)}y,$$

which, by definition, is

$$d_n^{f,E_{r'}}(x) = y.$$

Next, we prove the second part of the proposition. By Definition 5.4, the (f, E_r)-extension (6.27) is inessential if and only if there exists $0 < a \leq n - e(f)$ and

$$\lambda^a x' \in E_\infty^{s+a,t+a,t}(\nu X/\lambda^{r'-1}) \cong Z_{r'-1-a}^{s+a,t+a}(X)/B_{1+a}^{s+a,t+a}(X),$$

such that

$$d_{n-a}^{\hat{f}_{r'-1}}(\lambda^a x') = \lambda^{n-e(f)}y, \tag{6.28}$$

and this differential is not induced by (ρ_X, ρ_Y), as we assume that $d_n^{f,E_r}(x) = y$ is essential.

There are two scenarios where this differential is not induced by (ρ_X, ρ_Y). The first case occurs when $\lambda^a x'$ is not in the image of ρ_X at all, which is equivalent to $x' \notin Z_{r-1-a}^{s+a,t+a}(X)$. (See Figure 6.)

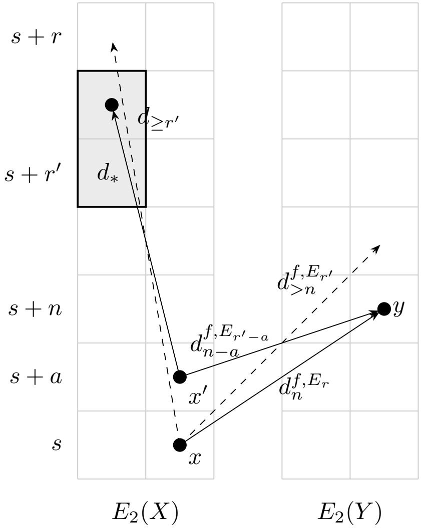

FIGURE 6. Extensions across pages

The second case occurs when $\lambda^a x'$ is in the image of ρ_X, but it supports an essential $\hat{f}_{r-1}$-extension that is strictly shorter than the differential (6.28).

To further explore this scenario, we replace x with $\lambda^a x'$ and analyze successive essential extensions. This process is repeated iteratively until the first case is reached. Ultimately, this leads to the existence of some $0 < a' \leq n - e(f)$ and an element

$$x'' \in Z^{s+a',t+a'}_{r'-1-a'}(X) \backslash Z^{s+a',t+a'}_{r-1-a'}(X)$$

such that

$$d^{\hat{f}_{r'-1}}_{n-a'-b}(\lambda^{a'} x'') = \lambda^{n-b-e(f)} y'$$

for some $b > 0$. This represents a crossing of $d^{\hat{f}_{r'-1}}_{n}(x) = \lambda^{n-e(f)} y$.

□

The following Corollary 6.29 is the contrapositive statement of the second part of Proposition 6.26.

Corollary 6.29. *Suppose $r \geq r'$ and there exists an $(f, E_{r'})$-extension*

$$d^{f,E_{r'}}_{n}(x) = y$$

for $x \in Z^{s,t}_{r-1}(X)$ and $y \in Z^{s+n,t+n}_{r'-1-n+e(f)}(Y)$. Assume that this extension has no crossing of the form $d^{f,E_{r'-a}}_{n-a-b}(x') = y'$ for any $b > 0$, $0 < a \leq n - e(f)$, and

$$x' \in Z^{s+a,t+a}_{r'-1-a}(X) \backslash Z^{s+a,t+a}_{r-1-a}(X).$$

Under these conditions, we also have an (f, E_r)-extension

$$d^{f,E_r}_{n}(x) = y.$$

Example 6.30. In Example **??**, we use the Generalized Mahowald Trick to establish the (f, E_3)-extension:

$$d^{f,E_3}_{2}(h_0 h_4^2) = h_0 p,$$

for the map $f = \nu : S^3 \to S^0$, as discussed in Example **??**(2). However, to apply the Generalized Leibniz Rule in proving the classical Adams differential

$$d_3(h_2 h_5) = h_0 p$$

in Example **??**, we need the E_∞-page version of the (f, E_3)-extension:

$$d^{f,E_\infty}_{2}(h_0 h_4^2) = h_0 p.$$

We apply Corollary 6.29 to confirm this.

As checked in Example **??**(2), the (f, E_3)-extension has no crossing. Consequently, by Corollary 6.29, we obtain the required (f, E_∞)-extension:

$$d^{f,E_\infty}_{2}(h_0 h_4^2) = h_0 p.$$

7. Proof of the main theorem

In this section, we present the proof of our main Theorem 7.1.

Theorem 7.1 (Theorem 7.1)**.** *The element h_6^2 survives to the E_∞-page in the Adams spectral sequence.*

We first recall the following theorem, known as the inductive method, originally by Barratt–Jones–Mahowald [1] and later extended to the $\mathrm{H}\mathbb{F}_2$-synthetic setting by Burklund–Xu [7, Proposition 7.19].

Notation 7.2. Let $\theta_5 = [h_5^2]$ represent any synthetic homotopy class in $\pi_{62,62+2}S^{0,0}$ detected by h_5^2 in the Adams E_2-page. For convenience, we use the same notation, θ_5, to denote its image in $\pi_{62,62+2}S^{0,0}/\lambda^r$ via the map $S^{0,0} \to S^{0,0}/\lambda^r$ for all $r \geq 1$. Similarly, let $\eta = [h_1] \in \pi_{1,1+1}S^{0,0}$.

Axiom 7.3 (Barratt–Jones–Mahowald, Burklund–Xu)**.**

(1) *The element h_6^2 survives to the E_{r+3}-page of the classical Adams spectral sequence if and only if for some θ_5,*

$$\lambda\eta\theta_5^2 = 0 \textit{ in } \pi_{125,125+4}S^{0,0}/\lambda^{r+1}.$$

(2) *In particular, h_6^2 is a permanent cycle in the classical Adams spectral sequence if and only if for some θ_5,*

$$\lambda\eta\theta_5^2 = 0 \textit{ in } \pi_{125,125+4}S^{0,0}.$$

Remark 7.4. The statement in Axiom 7.3(1) was originally stated as

$$\eta\theta_5^2 = 0 \text{ in } \pi_{125,125+5}S^{0,0}/\lambda^r$$

in [7, Proposition 7.19]. Upon inspection, $\pi_{125,125+5}S^{0,0}$ doesn't contain any λ-torsion classes, so this is equivalent to the version we stated, which is more consistent with the statement in part (2).

Remark 7.5. Axiom 7.3 is proved using the quadratic construction on a map from the mod 2 Moore spectrum to the sphere spectrum, where the restriction on the bottom cell is θ_5. Therefore, it is necessary to use a θ_5 of order 2.

Notably, [19, 10] confirms that all classical θ_5's indeed have order 2. Furthermore, from Proposition 0.121 and an analysis of the differentials in the classical Adams spectral sequence, we find that $\pi_{62,62+2}S^{0,0}$ doesn't contain any λ-torsion. Consequently, all synthetic θ_5's also have order 2, making it valid to apply Axiom 7.3 to any θ_5.

Additionally, since the proof of Axiom 7.3 shows that the expression $\lambda\eta\theta_5^2$ corresponds to the total differential $\delta_1 : S^{0,0}/\lambda \to S^{1,-1}$ on h_6^2, the value of the expression $\lambda\eta\theta_5^2$ is consistent for every choice of θ_5. (Note that our grading for the triangulation translation functor is smashing with $S^{1,0}$, which is consistent with [6, Appendix A] but is different from [7, Section 7], so the target of δ_1 is $S^{1,-1}$.)

We pay special attention to the following three elements on the classical Adams E_2-page (see Figure 7 and Tables 7, 9, 5, 6 in the Appendix):

$$\begin{aligned} h_1h_4x_{109,12} &\in \mathrm{Ext}_A^{14,125+14}, \\ x_{126,8,4} + x_{126,8} &\in \mathrm{Ext}_A^{8,126+8}, \\ h_0^2x_{124,8} &\in \mathrm{Ext}_A^{10,124+10}, \\ g^4\Delta h_1 g &\in \mathrm{Ext}_A^{25,125+25}. \end{aligned}$$

For the right side of Figure 7, we use dashed differentials to indicate the shortest possible nonzero differentials that these elements could support.

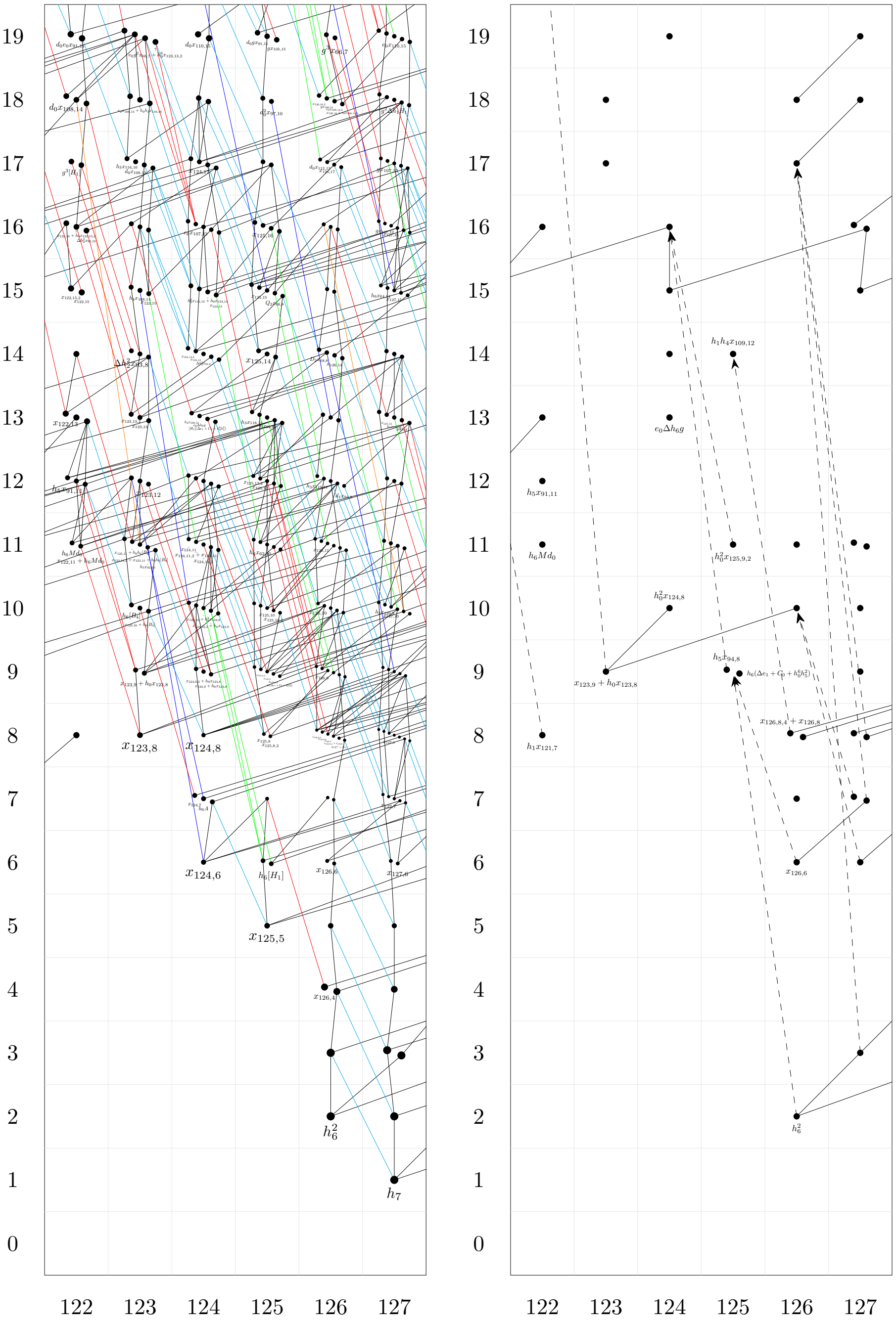

Figure 7. The Adams E_2 and E_∞-pages of S^0 near h_6^2

From Figure 7 and Tables 5, 9, 7, 6 in the Appendix, we know that

Axiom 7.6. *(1) $x_{126,8,4} + x_{126,8}$ survives to the E_6-page.*

(2) $h_1h_4x_{109,12}$ is a permanent cycle, and can only be killed by

$$d_6(x_{126,8,4} + x_{126,8}) \text{ or } d_{12}(h_6^2).$$

(3) $h_0^2x_{124,8}$ survives to the E_∞-page.

(4) In $\mathrm{Ext}_A^{25,125+25}$, the element $g^4\Delta h_1 g$ is the only one that survives to the classical E_5-page.

Remark 7.7. From Table 9 in the Appendix, we have

$$d_3(x_{126,6}) = h_5x_{94,8}, \text{ or } h_5x_{94,8} + h_6(\Delta e_1 + C_0 + h_0^6h_5^2) \neq 0.$$

Therefore, the element $x_{126,6} \in \mathrm{Ext}_A^{6,126+6}$ cannot kill $h_1h_4x_{109,12}$.

We will apply Axiom 7.3 to prove the following Proposition 7.8.

Proposition 7.8. *Exactly one of the following two statements is true:*

(1) The element h_6^2 survives to the E_∞-page.

(2) There is a nonzero classical Adams differential

$$d_{12}(h_6^2) = h_1h_4x_{109,12}.$$

Furthermore, statement (2) *is true if and only if the following three statements are all true:*

(3)
$$d_6(x_{126,8,4} + x_{126,8}) = 0.$$

(4) There exists a θ_5 such that θ_5^2 is detected by $\lambda^6h_0^2x_{124,8}$. In particular,

$$\theta_5^2 = \lambda^6[h_0^2x_{124,8}] \neq 0 \in \pi_{124,124+4}S^{0,0}$$

for some $[h_0^2x_{124,8}]$.

(5) There exists a homotopy class $[h_0^2x_{124,8}]$ such that $\lambda^3\eta[h_0^2x_{124,8}]$ is detected by $\lambda^6h_1h_4x_{109,12}$. In particular, we have

$$\lambda^3\eta[h_0^2x_{124,8}] = \lambda^6[h_1h_4x_{109,12}] \in \pi_{125,125+8}S^{0,0}$$

for some $[h_1h_4x_{109,12}]$.

By further analyzing classical Adams differentials, we reduce the η-extension in statement (5) of Proposition 7.8 to a specific 2-extension in stem 125 (Corollary 7.18), and then compare it with a particular ν-extension in stem 125 (Lemma 7.20) to demonstrate that the 2-extension cannot hold. This ultimately leads to the proof of Proposition 7.9.

Proposition 7.9. *If statement* (3) *is true, then statement* (5) *in Proposition 7.8 must be false.*

Proof of Theorem 7.1. From Proposition 7.9, at least one of statements (3) or (5) in Proposition 7.8 is false. Consequently, statement (2) is also false, which confirms that statement (1) in Proposition 7.8 is true. □

In the rest of this section, we prove Propositions 7.8 and 7.9.

To prove Proposition 7.8, we first establish Lemmas 7.10 and 7.11, which demonstrate that the existential statements (4) and (5) in Proposition 7.8 are equivalent to their corresponding universal statements (4′) and (5′).

Lemma 7.10. *The statement* (4) *in Proposition 7.8 is equivalent to the following statement* (4′)*:*

(4) *For every θ_5, we have θ_5^2 is detected by $\lambda^6 h_0^2 x_{124,8}$. In particular,*

$$\theta_5^2 = \lambda^6 [h_0^2 x_{124,8}] \neq 0 \in \pi_{124,124+4} S^{0,0}$$

for some $[h_0^2 x_{124,8}]$.

Proof. We only need to show that statement (4) implies statement (4′).

According to [10],

$$\pi_{62} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

and is generated by θ_5 and classes of $\mathrm{AF} = 6, 8, 10$. Therefore, the indeterminacy of the classical θ_5, or the difference of any two choices of classical θ_5, lies in $\mathrm{AF} \geq 6$.

As explained in Remark 7.5, $\pi_{62,62+2} S^{0,0}$ contains no λ-torsion, and therefore, the indeterminacy of the synthetic θ_5 also belongs to $\mathrm{AF} \geq 6$. Since every θ_5 has order 2, the indeterminacy of the synthetic θ_5^2 lies in $\mathrm{AF} \geq 12$.

Therefore, if for some θ_5, θ_5^2 is detected by this specific element $\lambda^6 h_0^2 x_{124,8}$, which is nonzero in $\mathrm{AF} = 10$, then for any θ_5, θ_5^2 is nonzero and detected by $\lambda^6 h_0^2 x_{124,8}$. □

Lemma 7.11. *The statement* (5) *in Proposition 7.8 is equivalent to the following statement* (5′)*:*

(5) *For every homotopy class $[h_0^2 x_{124,8}]$, we have $\lambda^3 \eta [h_0^2 x_{124,8}]$ is detected by $\lambda^6 h_1 h_4 x_{109,12}$. In particular, we have*

$$\lambda^3 \eta [h_0^2 x_{124,8}] = \lambda^6 [h_1 h_4 x_{109,12}] \in \pi_{125,125+8} S^{0,0}$$

for some $[h_1 h_4 x_{109,12}]$.

Proof. We only need to show that statement (5) implies statement (5′), which is sufficient to show that the indeterminacy of $[h_0^2 x_{124,8}]$, or the difference between any two homotopy classes $[h_0^2 x_{124,8}]$, when multiplied by $\lambda^3 \eta$, belongs to $\mathrm{AF} \geq 15$, given that $[h_1 h_4 x_{109,12}]$ has $\mathrm{AF} = 14$.

Since $[h_0^2 x_{124,8}]$ has $\mathrm{AF} = 10$, the indeterminacy is generated by cycles in $\mathrm{AF} \geq 11$ of stem 124. We observe that classical cycles in $\mathrm{AF} = 11, 12$ of stem 124 are all killed by Adams d_2 or d_3-differentials, and cycles in $\mathrm{AF} = 13$ are all annihilated by h_1 in Ext. Therefore, we conclude that the indeterminacy, when multiplied by $\lambda^3 \eta$, belongs to $\mathrm{AF} \geq 15$. □

Remark 7.12. From Axiom 7.6(2) and the rigidity Axioms 0.119 and 0.120 for the synthetic Adams spectral sequence for $S^{0,0}$, the right side of the equation in statement (5′) is nonzero if and only if statement (3) is true.

By Lemmas 7.10 and 7.11, we will freely interchange statements (4) and (4′), as well as (5) and (5′), depending on the context.

Now we prove Proposition 7.8.

Proof of Proposition 7.8. From the proof of Axiom 7.3 [7] we know that

$$\delta_1(h_6^2) = \lambda \eta \theta_5^2,$$

where δ_1 is the map $S^{0,0}/\lambda \to S^{1,-1}$. Suppose that $\eta\theta_5^2$ is detected by $\lambda^{n-5}T_n$ in $\mathrm{AF} = n$ of the synthetic Adams spectral sequence for some $T_n \in \mathrm{Ext}_A^{n,125+n}$. This implies a differential in the δ_1-ESS:

$$d_{n-2}^{\delta_1}(h_6^2) = \lambda^{n-2}T_n.$$

By Axioms 0.119, 0.120 and Corollary 4.10, this is equivalent to a synthetic Adams differential

$$d_{n-2}(h_6^2) = \lambda^{n-3}T_n$$

and a classical Adams differential

$$d_{n-2}(h_6^2) = T_n.$$

This differential is nonzero if and only if T_n is nonzero on the classical E_{n-2}-page, i.e., not the image of a differential $d_{\leq n-3}$. In particular, when $\lambda\eta\theta_5^2 = 0$, we conclude that h_6^2 is a permanent cycle.

We first show that statements (3), (4′) and (5′) together imply statement (2).

By statements (4′) and (5′), we have that for any θ_5,

$$\lambda\eta\theta_5^2 = \lambda\eta \cdot \lambda^6[h_0^2x_{124,8}] = \lambda^4 \cdot \lambda^3\eta[h_0^2x_{124,8}] = \lambda^{10}[h_1h_4x_{109,12}].$$

Recall from Axiom 7.6(2) that $h_1h_4x_{109,12}$ is a permanent cycle, and can only be killed classically by

$$d_6(x_{126,8,4} + x_{126,8}) \text{ or } d_{12}(h_6^2).$$

Therefore, statement (3) implies that the expression $\lambda^{10}[h_1h_4x_{109,12}]$ is nonzero in synthetic homotopy, leading to a nonzero classical d_{12}-differential:

$$d_{12}(h_6^2) = h_1h_4x_{109,12}.$$

We will complete the proof of Proposition 7.8 by showing that if one the statements (3), (4) or (5) is false, then statement (2) is also false, and in this case,

$$\eta\theta_5^2 = 0 \in \pi_{125,125+5}S^{0,0},$$

making statement (1) true. This conclusion is reached by estimating the Adams filtration of θ_5^2 and subsequently that of the expression $\eta\theta_5^2$.

We begin by estimating the Adams filtration of θ_5^2 in $\pi_{124,124+4}S^{0,0}$. First, we observe that this group $\pi_{124,124+4}S^{0,0}$ does not contain any λ-torsion classes. This is because, in the 125-stem, the group (from Table 7 in the Appendix)

$$\mathrm{Ext}_A^{i,125+i} = 0 \text{ for } i \leq 4,$$

and thus, by the rigidity Axioms 0.119 and 0.120 for the synthetic Adams spectral sequence for $S^{0,0}$, in the 124-stem, λ-torsion classes can only appear in $\pi_{124,124+j}S^{0,0}$ for $j \geq 7$.

Additionally, by analyzing classical Adams differentials, the Adams filtration of θ_5^2 is at least 10. In the case where it is of Adams filtration 10, it is detected by the element $h_0^2x_{124,8}$, which, according to Axiom 7.6(3), is a permanent cycle and cannot be killed.

Upon further inspection of the differentials associated with elements in stem 124 and filtration between 10 and 13, we are left with three possibilities:

1. $\theta_5^2 = \lambda^6 \cdot [h_0^2x_{124,8}]$, which is statement (4), or
2. $\theta_5^2 = \lambda^9 \cdot [e_0\Delta h_6g]$, where $e_0\Delta h_6g$ is permanent cycle in $\mathrm{AF} = 13$, or
3. θ_5^2 is a λ^{10}-multiple.

Since in Ext, we have $h_1 \cdot e_0\Delta h_6 g = 0$, we deduce that in either possibilities (b) or (c), $\eta\theta_5^2$ is a λ^{10}-multiple. In other words, $\eta\theta_5^2$ has AF ≥ 15. In the group $\pi_{125,125+5}S^{0,0}$, the only class that is a λ^{10}-multiple is actually λ-free: By Axiom 7.6(4), it is $\lambda^{20}g^4\Delta h_1 g$ in AF $= 25$ and is detected by tmf (see [3] for the Hurewicz image of tmf). Since θ_5 maps to zero in tmf, we have that $\eta\theta_5^2$ maps to zero in tmf and therefore must be zero in this case.

This shows that if statement (4) is false, then $\eta\theta_5^2 = 0$, and therefore h_6^2 is a permanent cycle.

Hence, we focus on the remaining possibility (a), assuming statement (4) holds:

$$\theta_5^2 = \lambda^6 \cdot [h_0^2 x_{124,8}].$$

By the proof of Lemma 7.11, the only nonzero possibility for $\lambda^3\eta[h_0^2 x_{124,8}]$ is $\lambda^6[h_1 h_4 x_{109,12}]$. Therefore, if statement (5) is false, then using the same tmf detection argument, we conclude that $\eta\theta_5^2 = 0$, and consequently, h_6^2 is a permanent cycle.

Finally, if statement (3) is false, an inspection reveals the only alternative classical differential:

$$d_6(x_{126,8,4} + x_{126,8}) = h_1 h_4 x_{109,12}.$$

This, combined with the tmf detection argument, would again imply $\eta\theta_5^2 = 0$, and consequently, h_6^2 is a permanent cycle. This completes the proof. □

Before we prove Proposition 7.9, we first state and prove a few lemmas.

For Lemma 7.14, we draw attention to the following element in Ext:

$$x_{123,9} + h_0 x_{123,8} \in \mathrm{Ext}_A^{9,123+9}.$$

From Tables 3, 7 in the Appendix, we have

Axiom 7.13. *(1) $x_{123,9} + h_0 x_{123,8}$ survives to the Adams E_{12}-page, and is not killed by any classical differential.*

(2) We have a classical nonzero differential

$$d_2(x_{125,8}) = h_1(x_{123,9} + h_0 x_{123,8}) + h_0^2 x_{124,8}.$$

Lemma 7.14. *There exists a homotopy class*

$$\alpha_1 = [x_{123,9} + h_0 x_{123,8}] \in \pi_{123,123+9}S^{0,0}/\lambda^9$$

with the following properties:

(1) For any homotopy class $[h_0^2 x_{124,8}]$, there exist homotopy classes

$$\alpha_2 \in \pi_{124,124+13}S^{0,0}/\lambda^9,\ \alpha_3 \in \pi_{125,125+15}S^{0,0}/\lambda^9,$$

such that

$$\begin{aligned} \lambda^3\eta \cdot \alpha_1 &= \lambda^3[h_0^2 x_{124,8}] + \lambda^6\alpha_2 && \in \pi_{124,124+7}S^{0,0}/\lambda^9, \\ \eta \cdot \alpha_2 &= \lambda \cdot \alpha_3 && \in \pi_{125,125+14}S^{0,0}/\lambda^9, \end{aligned}$$

(2)
$$\lambda^3 \cdot \alpha_1 \cdot [h_0] = 0 \in \pi_{123,123+7}S^{0,0}/\lambda^9.$$

Proof. From Axiom 7.13(1), the element $x_{123,9} + h_0 x_{123,8}$ survives to the Adams E_{12}-page, and is not killed by any classical differential.

Let $\alpha_1 = [x_{123,9} + h_0x_{123,8}] \in \pi_{123,123+9}S^{0,0}/\lambda^{11}$ denote any homotopy class detected by $x_{123,9} + h_0x_{123,8}$. For simplicity, we also use α_1 to denote its images in $\pi_{123,123+9}S^{0,0}/\lambda^r$ for $1 \leq r \leq 10$, under the following sequence of maps:

$$\begin{tikzcd}
S^{0,0}/\lambda^{11} \arrow[r] & S^{0,0}/\lambda^{10} \arrow[r] & \cdots \arrow[r] & S^{0,0}/\lambda \\
\alpha_1 = [x_{123,9} + h_0x_{123,8}] \arrow[r, "\mapsto"] & \alpha_1 \arrow[r, "\mapsto"] & \cdots \arrow[r, "\mapsto"] & x_{123,9} + h_0x_{123,8}
\end{tikzcd}$$

From the nonzero d_2-differential in Axiom 7.13, we have

$$h_1 \cdot (x_{123,9} + h_0x_{123,8}) = h_0^2x_{124,8},$$

on the classical E_3-page. This implies synthetically, for $2 \leq r \leq 11$,

$$\lambda\eta \cdot \alpha_1 + \lambda[h_0^2x_{124,8}] \in \pi_{124,124+9}S^{0,0}/\lambda^r$$

lies in AF ≥ 11 for any $[h_0^2x_{124,8}]$.

By inspection, as in the proof of Lemma 7.11,

$$\lambda^3\eta \cdot \alpha_1 + \lambda^3[h_0^2x_{124,8}] \in \pi_{124,124+7}S^{0,0}/\lambda^9$$

has AF ≥ 13. The only possibility for it to have AF $= 13$ is that it is detected by the element $\lambda^6e_0\Delta h_6g$. (Note that the class $[\lambda^6h_4x_{109,12}]$ is irrelevant due to the nonzero differential $d_3(\lambda^6h_4x_{109,12}) = \lambda^8h_1x_{122,15,2}$.) Since in Ext we have

$$h_1 \cdot e_0\Delta h_6g = 0,$$

we may choose $\alpha_2 \in \pi_{124,124+13}S^{0,0}/\lambda^9$ to be any class detected by $\lambda^6e_0\Delta h_6g$, with the property that $\eta\alpha_2$ is λ-divisible. Thus, there exists an α_3 such that $\eta\alpha_2 = \lambda\alpha_3$.

This proves the required property (1).

For the relation in property (2), we will first prove it in $\pi_{123,123+7}S^{0,0}/\lambda^{11}$ and then map it to $\pi_{123,123+7}S^{0,0}/\lambda^9$.

By Proposition 0.122 for the synthetic Adams spectral sequence for $S^{0,0}/\lambda^{11}$, the expression

$$\lambda^3 \cdot \alpha_1 \cdot [h_0] \in \pi_{123,123+7}S^{0,0}/\lambda^{11}$$

has AF ≥ 17. In particular, the values

$$\lambda^4(x_{123,11,2} + x_{123,11} + h_0h_6B_4) \text{ in AF} = 11,$$

$$\lambda^8h_0^2x_{123,13,2} \text{ in AF} = 15$$

can be ruled out due to the nonzero Adams differentials

$$d_7(\lambda^4 \cdot (x_{123,11,2} + x_{123,11} + h_0h_6B_4)\ ) = \lambda^{10}h_1x_{121,17},$$

$$d_3(\lambda^8 \cdot h_0^2x_{123,13,2}) = \lambda^{10}h_0^2x_{122,16},$$

in the synthetic Adams spectral sequence for $S^{0,0}/\lambda^{11}$, which are zero in the spectral sequence for $S^{0,0}/\lambda^9$.

Therefore, the expression $\lambda^3 \cdot \alpha_1 \cdot [h_0]$ is λ^{10}-divisible in $\pi_{123,123+7}S^{0,0}/\lambda^{11}$. Mapping it further to $\pi_{123,123+7}S^{0,0}/\lambda^9$, we conclude that it is zero. □

For Lemma 7.16, we draw attention to the following element in Ext:

$$h_0^2x_{125,9,2} \in \mathrm{Ext}_A^{11,125+11}.$$

From Table 7 in the Appendix, we have

Axiom 7.15. *The element $h_0^2x_{125,9,2}$ survives to the Adams E_5-page, and is not killed by any classical differential.*

Lemma 7.16. *Assuming that both statements* (3) *and* (5′) *in Proposition 7.8 are true, the synthetic Toda bracket*

$$\langle \lambda^3\alpha_1, [h_0], \eta\rangle \subset \pi_{125,125+7}S^{0,0}/\lambda^9$$

does not contain zero, and is detected by $\lambda^4 h_0^2 x_{125,9,2}$*. Here* $\alpha_1 = [x_{123,9} + h_0 x_{123,8}]$ *refers to the homotopy class described in Lemma 7.14.*

Note that the synthetic Toda bracket in Lemma 7.16 is well defined, as the homotopy class α_1 in Lemma 7.14 satisfies the relation $\lambda^3\alpha_1 \cdot [h_0] = 0$.

Remark 7.17. According to Axiom 7.15, it is not yet known whether the element $h_0^2 x_{125,9,2}$ supports a nonzero d_5-differential. Assuming that both statements (3) and (5′) in Proposition 7.8 are true, Lemma 7.16 specifically implies that $\lambda^4 h_0^2 x_{125,9,2}$ detects a nonzero homotopy class in $\pi_{125,125+7}S^{0,0}/\lambda^9$. Therefore, under these assumptions, we would have $d_5(h_0^2 x_{125,9,2}) = 0$.

Proof of Lemma 7.16. We assume both statements (3) and (5′) in Proposition 7.8 are true. From statement (5′), we have

$$\lambda^3\eta[h_0^2 x_{124,8}] = \lambda^6[h_1 h_4 x_{109,12}] \in \pi_{125,125+8}S^{0,0}.$$

Mapping this relation to $S^{0,0}/\lambda^9$, and applying the following relations from Lemma 7.14

$$\begin{aligned} \lambda^3\eta \cdot \alpha_1 &= \lambda^3[h_0^2 x_{124,8}] + \lambda^6\alpha_2 && \in \pi_{124,124+7}S^{0,0}/\lambda^9, \\ \eta \cdot \alpha_2 &= \lambda \cdot \alpha_3 && \in \pi_{125,125+14}S^{0,0}/\lambda^9, \end{aligned}$$

we have

$$\begin{aligned} \eta \cdot \lambda^3\eta\alpha_1 &= \eta \cdot \lambda^3[h_0^2 x_{124,8}] + \eta \cdot \lambda^6\alpha_2 \\ &= \lambda^6[h_1 h_4 x_{109,12}] + \lambda^7\alpha_3 \in \pi_{125,125+8}S^{0,0}/\lambda^9, \end{aligned}$$

which, from statement (3) and Remark 7.17, is nonzero and detected by $\lambda^6 h_1 h_4 x_{109,12}$ in AF $= 14$.

On the other hand, since $\eta^2 = \langle [h_0], \eta, [h_0]\rangle$, we have

$$\eta \cdot \lambda^3\eta\alpha_1 = \lambda^3\alpha_1 \cdot \langle [h_0], \eta, [h_0]\rangle = \langle \lambda^3\alpha_1, [h_0], \eta\rangle \cdot [h_0].$$

Therefore, the synthetic Toda bracket $\langle \lambda^3\alpha_1, [h_0], \eta\rangle$ does not contain zero, and its $[h_0]$-multiple is detected by $\lambda^6 h_1 h_4 x_{109,12}$ in AF $= 14$. Since $h_1 h_4 x_{109,12}$ is not h_0-divisible in Ext, the synthetic Toda bracket is detected by an element in AF ≤ 12.

This synthetic Toda bracket $\langle \lambda^3\alpha_1, [h_0], \eta\rangle$ lies in $\pi_{125,125+7}S^{0,0}/\lambda^9$, whose AF $\leq$ 12 part is generated by

$$\begin{aligned} [h_0^2 x_{125,5}] &\text{ in AF} = 7, \\ \lambda^2[h_6(\Delta e_1 + C_0 + h_0^6 h_5^2)] &\text{ in AF} = 9, \\ [\lambda^4 h_0^2 x_{125,9,2}] &\text{ in AF} = 11. \end{aligned}$$

From Table 9 in the Appendix, we have

$$0 \neq d_3(x_{126,6}) = \lambda^2 h_5 x_{94,8} + \text{ possibly } \lambda^2 h_6(\Delta e_1 + C_0 + h_0^6 h_5^2).$$

In both scenarios, $\lambda^2[h_6(\Delta e_1 + C_0 + h_0^6 h_5^2)]$ remains and is in AF $= 9$.

For the rest of the proof, we only need to rule out the cases $[h_0^2 x_{125,5}]$ in AF $= 7$ and $\lambda^2[h_6(\Delta e_1 + C_0 + h_0^6 h_5^2)]$ in AF $= 9$.

For $[h_0^2x_{125,5}]$ in $\mathrm{AF}=7$, due to the nonzero d_3-differential

$$d_3(x_{126,4}) = \lambda^2h_0^2x_{125,5},$$

it can be chosen to be annihilated by λ^2.

However, from statement (3) and Remark 7.17, $\lambda^8[h_1h_4x_{109,12}]$ remains nonzero in the homotopy of $S^{0,0}/\lambda^9$. Therefore, the synthetic Toda bracket is not annihilated by λ^2. As discussed earlier, its $[h_0]$-multiple is detected by $\lambda^6h_1h_4x_{109,12}$, and thus, this case of $[h_0^2x_{125,5}]$ can be ruled out.

For $\lambda^2[h_6(\Delta e_1+C_0+h_0^6h_5^2)]$ in $\mathrm{AF}=9$, we first consider a classical Toda bracket in stem 125:

$$\langle\theta_5, 2, [\Delta e_1 + C_0 + h_0^6h_5^2]\rangle.$$

From [19, 10], $\pi_{62} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$, so in particular both θ_5 and $[\Delta e_1 + C_0 + h_0^6h_5^2]$ have order 2, and this classical Toda bracket is well defined.

From classical d_2-differentials:

$$d_2(h_6) = h_0h_5^2,\ \ d_2(h_0^6h_6) = h_0(\Delta e_1 + C_0 + h_0^6h_5^2),$$

we obtain the following Massey product on the E_3-page

$$h_6(\Delta e_1 + C_0 + h_0^6h_5^2) = \langle h_5^2, h_0, \Delta e_1 + C_0 + h_0^6h_5^2\rangle,$$

and we check that it has zero indeterminacy. Further analysis shows that there are no crossing differentials, as per the criteria of Moss's theorem [18, Theorem 1.2] (noting that crossing differentials in Moss's theorem have a different meaning from our definition). Therefore, we have a classical Toda bracket

$$[h_6(\Delta e_1 + C_0 + h_0^6h_5^2)] \in \langle\theta_5, 2, [\Delta e_1 + C_0 + h_0^6h_5^2]\rangle.$$

Synthetically, by inspection, we also have $2\theta_5 = 0$ and $2[\Delta e_1 + C_0 + h_0^6h_5^2] = 0$. It follows that there is a corresponding synthetic Toda bracket

$$[h_6(\Delta e_1 + C_0 + h_0^6h_5^2)] \in \langle\theta_5, 2, [\Delta e_1 + C_0 + h_0^6h_5^2]\rangle.$$

Multiplying by $\lambda^2[h_0]$, we get:

$$\begin{aligned}\lambda^2[h_0]\cdot[h_6(\Delta e_1 + C_0 + h_0^6h_5^2)] &= \lambda^2[h_0]\cdot\langle\theta_5, 2, [\Delta e_1 + C_0 + h_0^6h_5^2]\rangle\\ &= \lambda\cdot\langle 2, \theta_5, 2\rangle[\Delta e_1 + C_0 + h_0^6h_5^2]\\ &= \lambda^3\eta\theta_5[\Delta e_1 + C_0 + h_0^6h_5^2].\end{aligned}$$

Note that all expressions in the above equation have zero indeterminacy.

If the synthetic Toda bracket $\langle\lambda^3\alpha_1, [h_0], \eta\rangle$ were $\lambda^2[h_6(\Delta e_1+C_0+h_0^6h_5^2)]$, mapping the above equation to $S^{0,0}/\lambda^9$, we would then have a nonzero equation in $\pi_{125,125+8}S^{0,0}/\lambda^9$.

$$\begin{aligned}\lambda^3\eta[h_0^2x_{124,8}] = \lambda^6[h_1h_4x_{109,12}] &= \langle\lambda^3\alpha_1, [h_0], \eta\rangle[h_0] + \lambda^6\eta\alpha_2\\ &= \lambda^2[h_6(\Delta e_1 + C_0 + h_0^6h_5^2)][h_0] + \lambda^6\eta\alpha_2\\ &= \lambda^3\eta\theta_5[\Delta e_1 + C_0 + h_0^6h_5^2] + \lambda^6\eta\alpha_2.\end{aligned}$$

For this equation to be nonzero, we must have a nonzero equation

$$\lambda^3[h_0^2x_{124,8}] = \lambda^3\theta_5[\Delta e_1 + C_0 + h_0^6h_5^2] + \lambda^6\alpha_2 \ \text{ in } \ \pi_{124,124+7}S^{0,0}/\lambda^9.$$

However, in Ext, we have

$$h_5^2(\Delta e_1 + C_0 + h_0^6h_5^2) = 0 \neq h_0^2x_{124,8} \in \mathrm{Ext}_A^{10,124+10},$$

so this equation is not possible.

We have ruled out the possibility that the synthetic Toda bracket $\langle \lambda^3\alpha_1, [h_0], \eta\rangle$ is detected by $\lambda^2[h_6(\Delta e_1 + C_0 + h_0^6h_5^2)]$ or $[h_0^2x_{125,5}]$. Therefore, we conclude that it must be detected by $[\lambda^4h_0^2x_{125,9,2}]$. □

From the proof of Lemma 7.16, we have the following $[h_0]$-extension.

Corollary 7.18. *Assuming that both statements* (3) *and* (5′) *in Proposition 7.8 are true, we have a relation: For any homotopy class* $[\lambda^4h_0^2x_{125,9,2}]$,

$$[\lambda^4h_0^2x_{125,9,2}] \cdot [h_0] = \lambda^6[h_1h_4x_{109,12}] \neq 0 \in \pi_{125,125+8}S^{0,0}/\lambda^9,$$

for some $[h_1h_4x_{109,12}]$.

Proof. From the proof of Lemma 7.16, there exists a homotopy class $[\lambda^4h_0^2x_{125,9,2}]$ contained in the synthetic Toda bracket $\langle\lambda^3\alpha_1, [h_0], \eta\rangle$, and we have

$$[\lambda^4h_0^2x_{125,9,2}] \cdot [h_0] = \langle\lambda^3\alpha_1, [h_0], \eta\rangle \cdot [h_0] = \lambda^6[h_1h_4x_{109,12}] + \lambda^7\alpha_3.$$

Since $\lambda^7\alpha_3$ is in a strictly higher filtration than $\lambda^6[h_1h_4x_{109,12}]$, we may choose another class $[h_1h_4x_{109,12}]$ such that the right-hand side of the above equation is simply $\lambda^6[h_1h_4x_{109,12}]$. From the discussion of this synthetic Toda bracket in the proof of Lemma 7.16, we know that the difference of any two classes detected by $\lambda^4h_0^2x_{125,9,2}$, when multiplied by $[h_0]$, is not detected by $\lambda^6[h_1h_4x_{109,12}]$. Therefore, the corollary holds for any choice of $[\lambda^4h_0^2x_{125,9,2}]$. □

We have one more Lemma 7.20 before we prove Proposition 7.9.

We draw attention to the following element in Ext:

$$h_1x_{121,7} \in \mathrm{Ext}_A^{8,122+8}.$$

From Table 2 in the Appendix, we have

Axiom 7.19. *The element* $h_1x_{121,7}$ *survives to the Adams* E_6*-page, and is not killed by any classical differential.*

Lemma 7.20. *There exists a homotopy class* $[\lambda^4h_1x_{121,7}] \in \pi_{122,122+4}S^{0,0}/\lambda^9$*, such that*

$$[\lambda^4h_1x_{121,7}] \cdot [h_2] = \lambda[\lambda^5h_0^2x_{125,9,2}] \in \pi_{125,125+5}S^{0,0}/\lambda^9.$$

Proof. From Axiom 7.19, we know that the element $h_1x_{121,7}$ may only support a nonzero d_r-differential for $r \geq 6$. By Proposition 4.6, $\lambda^4h_1x_{121,7}$ detects nonzero homotopy classes in $\pi_{122,122+4}S^{0,0}/\lambda^9$, and hence the required existence of such a homotopy class.

For the desired relation, we apply the Generalized Mahowald Trick Theorem 6.18. Consider the distinguished triangle

$$S^3 \xrightarrow{\nu} S^0 \xrightarrow{i} S^0/\nu \xrightarrow{q} S^4.$$

The short exact sequence on $\mathrm{H}\mathbb{F}_2$-homology

$$0 \to \mathrm{H}\mathbb{F}_{2*}S^0 \xrightarrow{i_*} \mathrm{H}\mathbb{F}_{2*}S^0/\nu \xrightarrow{q_*} \mathrm{H}\mathbb{F}_{2*}S^4 \to 0,$$

induces a long exact sequence on Ext-groups

$$\cdots \xrightarrow{\cdot h_2} \mathrm{Ext}_A^{*,*}(S^0) \xrightarrow{i_*} \mathrm{Ext}_A^{*,*}(S^0/\nu) \xrightarrow{q_*} \mathrm{Ext}_A^{*,*}(S^4) \xrightarrow{\cdot h_2} \cdots .$$

Also recall for notations, if an element x in $\mathrm{Ext}_A^{*,*}(S^0/\nu)$ satisfies

$$q_*(x) = a \neq 0 \in \mathrm{Ext}_A^{*,*}(S^4),$$

we denote x by $a[4]$; otherwise, due to exactness,

$$x = i_*(b) \text{ for some } b \in \mathrm{Ext}_A^{*,*}(S^0),$$

and in this case, we denote x by $b[0]$.

We consider

$$\begin{aligned} x &= h_1 x_{121,7} && \in \mathrm{Ext}_A^{8,130}, \\ \bar{x} &= h_1 x_{121,7}[4] + x_{126,8}[0] + x_{126,8,2}[0] && \in \mathrm{Ext}_A^{8,134}(S^0/\nu), \\ y &= h_0^2 x_{125,9,2} && \in \mathrm{Ext}_A^{11,136}, \\ \bar{y} &= h_0^2 x_{125,9,2}[0] && \in \mathrm{Ext}_A^{11,136}(S^0/\nu). \end{aligned}$$

For conditions in Theorem 6.18, we have:

1. $d_0^{q,E_2}(h_1 x_{121,7}[4] + x_{126,8}[0] + x_{126,8,2}[0]) = h_1 x_{121,7}$.
2. $d_3(h_1 x_{121,7}[4] + x_{126,8}[0] + x_{126,8,2}[0]) = h_0^2 x_{125,9,2}[0]$. This is a classical Adams d_3-differential for S^0/ν, and is obtained from our computations.
3. (a) The differential in (1) has no crossing, as it is a d_0-differential, and
   (b) Upon inspection, the differential in (2) has no crossing.
4. $d_0^{i,E_2}(h_0^2 x_{125,9,2}) = h_0^2 x_{125,9,2}[0]$, as $h_0^2 x_{125,9,2}$ is not divisible by h_2 in Ext.

Since all conditions of Theorem 6.18 are satisfied, we conclude that there is an $([h_2], E_4)$-extension:

$$d_3^{[h_2],E_4}(h_1 x_{121,7}) = h_0^2 x_{125,9,2}.$$

In other words, we have the following relation:

$$[h_1 x_{121,7}] \cdot [h_2] = \lambda^2 [h_0^2 x_{125,9,2}] \in \pi_{125,125+9} S^{0,0}/\lambda^3.$$

Using the map $\rho : S^{0,0}/\lambda^5 \to S^{0,0}/\lambda^3$ from Notation 4.1, we lift the above relation and obtain:

$$[h_1 x_{121,7}] \cdot [h_2] = [\lambda^2 h_0^2 x_{125,9,2}] \in \pi_{125,125+9} S^{0,0}/\lambda^5.$$

In fact, since $h_1 x_{121,7} \cdot h_2 = 0$ in AF $= 9$ of Ext, we might obtain a relation of the form:

$$[h_1 x_{121,7}] \cdot [h_2] = [\lambda x] + [\lambda^2 h_0^2 x_{125,9,2}] \in \pi_{125,125+9} S^{0,0}/\lambda^5,$$

for some element x in AF $= 10$. However, upon inspection, for all $x \in \mathrm{Ext}_A^{10,125+10}$, the homotopy class $[\lambda x]$ either does not exist or can be chosen to be zero.

Using the map $\lambda^4 : \Sigma^{0,-4} S^{0,0}/\lambda^5 \to S^{0,0}/\lambda^9$ from Notation 4.1, we further push the above relation and obtain the following relation:

$$[\lambda^4 h_1 x_{121,7}] \cdot [h_2] = \lambda [\lambda^5 h_0^2 x_{125,9,2}] \in \pi_{125,125+5} S^{0,0}/\lambda^9.$$

This completes the proof. □

Now we prove Proposition 7.9.

We draw attention to the following elements in Ext:

$$h_6 M d_0 \in \mathrm{Ext}_A^{11,122+11},$$

$$h_5 x_{91,11} \in \mathrm{Ext}_A^{12,122+12}.$$

From Table 2 in the Appendix, we have

Axiom 7.21.

(1) The element $h_6 M d_0$ survives to the Adams E_∞-page.

(2) The element $h_5 x_{91,11}$ survives to the Adams E_∞-page.

Proof of Proposition 7.9. We assume that statement (3) in Proposition 7.8 is true. For the sake of a contradiction, we also assume statement (5′) is true.

From Lemma 7.20 and Corollary 7.18, there exists a homotopy class $[\lambda^4 h_1 x_{121,7}] \in \pi_{122,122+4} S^{0,0}/\lambda^9$, such that

$$\begin{aligned}[\lambda^4 h_1 x_{121,7}] \cdot [h_2] \cdot [h_0] &= \lambda[\lambda^5 h_0^2 x_{125,9,2}] \cdot [h_0] \\ &= \lambda^8 [h_1 h_4 x_{109,12}] \neq 0 \in \pi_{125,125+6} S^{0,0}/\lambda^9,\end{aligned}$$

for some $[h_1 h_4 x_{109,12}]$. Note that statement (3) in Proposition 7.8, Axiom 7.6(2), and Proposition 4.6 imply that the expression $\lambda^8[h_1 h_4 x_{109,12}]$ is nonzero in the homotopy of $S^{0,0}/\lambda^9$.

Since the element detecting $h_1 h_4 x_{109,12}$ in not an h_2-multiple in Ext, we must have the expression

$$[\lambda^4 h_1 x_{121,7}] \cdot [h_0]$$

in $\pi_{122,122+5} S^{0,0}/\lambda^9$ be nonzero, and have $\mathrm{AF} \leq 12$. Upon inspection, the only possibilities are:

$$\lambda^6[h_6 M d_0] \text{ in } \mathrm{AF} = 11, \quad \lambda^7[h_5 x_{91,11}] \text{ in } \mathrm{AF} = 12.$$

From Axiom 7.21 both $h_6 M d_0$ and $h_5 x_{91,11}$ are nonzero permanent cycles in the classical Adams spectral sequence, so either possibility would lift to a relation in the homotopy groups of $S^{0,0}$.

For the case of $\lambda^6[h_6 M d_0]$, consider the expression

$$\lambda^2[h_6 M d_0] \cdot [h_2] \in \pi_{125,125+10} S^{0,0}.$$

Since $h_6 M d_0 \cdot h_2$ in $\mathrm{AF} = 12$ is killed by a classical d_2-differential, we know that

$$\lambda[h_6 M d_0] \cdot [h_2] \in \pi_{125,125+11} S^{0,0}$$

is in $\mathrm{AF} \geq 13$. Upon inspection, the permanent cycles in $\mathrm{Ext}_A^{13,125+13}$ are all killed by d_2 or d_4-differentials, and thus,

$$\lambda^2[h_6 M d_0] \cdot [h_2] \in \pi_{125,125+10} S^{0,0}$$

is in $\mathrm{AF} \geq 14$. Therefore, for the above relation in the homotopy of $S^{0,0}/\lambda^9$ to hold, we must have

$$\lambda^2[h_6 M d_0] \cdot [h_2] = \lambda^4[h_1 h_4 x_{109,12}] \in \pi_{125,125+10} S^{0,0}.$$

For the case of $\lambda^7[h_5 x_{91,11}]$, consider the expression

$$\lambda[h_5 x_{91,11}] \cdot [h_2] \in \pi_{125,125+12} S^{0,0}.$$

Since $h_5 x_{91,11} \cdot h_2$ in $\mathrm{AF} = 13$ is killed by a classical d_2-differential, we know that

$$\lambda[h_5 x_{91,11}] \cdot [h_2] \in \pi_{125,125+12} S^{0,0}$$

is in $\mathrm{AF} \geq 14$. Therefore, for the above relation in the homotopy of $S^{0,0}/\lambda^9$ to hold, we must have

$$\lambda[h_5 x_{91,11}] \cdot [h_2] = \lambda^2[h_1 h_4 x_{109,12}] \in \pi_{125,125+12} S^{0,0}.$$

In both cases, $\lambda^4[h_1 h_4 x_{109,12}]$ is a $\lambda[h_2]$-multiple in the homotopy of $S^{0,0}$. Since the classical $\nu \in \pi_3$ has $\mathrm{AF} = 1$, we have

$$S^{0,0}/(\lambda[h_2]) \simeq \nu(S^0/\nu).$$

By the rigidity Axiom 0.119 of the synthetic Adams spectral sequence for $S^{0,0}/(\lambda[h_2])$, we know that the element $\lambda^4 h_1 h_4 x_{109,12}[0]$ must be killed by a synthetic Adams differential, which corresponds to to a statement that in the classical Adams spectral sequence of S^0/ν, the element $h_1h_4x_{109,12}[0]$ must be killed by a d_r-differential for $r \leq 5$.

s	Elements	d_r	value
14	$h_0^{12}h_6^2[0]$	d_2^{-1}	$h_0^{11}h_7[0]$
	$x_{126,14}[0]$	d_3^{-1}	$(((x_{123,11,2}) + (x_{123,11}) + h_0h_6[B_4])[4])$
	$Q_2D_2(h_3[4]) + D_2x_{68,8}[0]$	d_3^{-1}	$(((x_{123,11,2}) + h_5(x_{92,10}))[4])$
	$D_2x_{68,8}[0]$	d_2	$h_0Q_2x_{68,8}[0]$
13	$h_0^{11}h_6^2[0]$	d_2^{-1}	$h_0^{10}h_7[0]$
	$h_1x_{120,11}(h_1[4])$	d_{12}	?
	$h_1x_{125,12,2}[0]$	d_5	$d_0^2x_{97,10}[0]$
	$h_6x_{56,10}(h_0h_2[4])$	d_3	$h_1x_{124,15}[0]$
	$(((x_{122,13}) + h_1^2(x_{120,11}) + h_0^2h_6(Md_0))[4])$	d_2	$x_{125,15}[0] + h_0^3x_{125,12}[0]$
	$h_0h_3x_{119,11}[0]$	d_2	$h_0^3x_{125,12}[0]$
12	$d_1x_{94,8}[0]$	d_2^{-1}	$x_{127,10}[0]$
	$h_0^{10}h_6^2[0]$	d_2^{-1}	$h_0^9h_7[0]$
	$((h_5(x_{91,11}) + h_0(x_{122,11}))[4])$	d_3	$Q_2x_{68,8}[0]$
	$h_3x_{119,11}[0]$	d_2	$h_0^2x_{125,12}[0]$
11	$h_0^9h_6^2[0]$	d_2^{-1}	$h_0^8h_7[0]$
	$x_{126,11}[0]$	d_3^{-1}	$x_{127,8}[0]$
	$h_1x_{125,10,2}[0] + h_1x_{125,10}[0]$		Permanent
	$h_1x_{125,10}[0]$	d_{14}	?
10	$h_0^2x_{126,8}[0]$	d_2^{-1}	$h_0x_{127,7}[0]$
	$h_0^8h_6^2[0]$	d_2^{-1}	$h_0^7h_7[0]$
	$x_{126,10}[0]$	d_3	$nx_{94,8}[0]$
9	$h_0x_{126,8}[0]$	d_2^{-1}	$x_{127,7}[0]$
	$h_0^7h_6^2[0]$	d_2^{-1}	$h_0^6h_7[0]$
	$h_1x_{125,8}[0]$	d_{16}	?
	$h_0x_{126,8,3}[0]$	d_4	$h_0x_{125,12}[0]$
	$x_{126,9}[0]$	d_3	$h_0^4x_{125,8}[0]$

TABLE 1. The classical Adams spectral sequence of S^0/ν for $9 \leq s \leq 14$ in stem 126

However, from Table 1 (obtained from [13] [12], and can be visualized from [11]), in the classical Adams spectral sequence of S^0/ν, the element $h_1h_4x_{109,12}[0]$ is not killed by any d_r for $r \leq 5$ (from the range of $9 \leq s \leq 14$ in stem 126). Therefore, we arrive at a contradiction. ☐

8. APPENDIX: THE CLASSICAL ADAMS SPECTRAL SEQUENCE IN THE RANGE $122 \leq t - s \leq 127, s \leq 25$

We provide a brief overview of Lin's computer program for computing Adams differentials and extensions. The program's functionality for propagating differentials and extensions relies on the following data:

- The Adams E_2-pages of a collection of CW spectra.
- Maps between these E_2-pages.
- Adams d_2-differentials for certain CW spectra.
- There are three manually added differentials: $d_5h_0^{24}h_6 = h_0^2P^6d_0$ and $d_6h_0^{55}h_7 = h_0^2x_{126,60}$ in the Adams spectral sequence of S^0 (from the image

of J), and $d_3 v_2^{16} = \beta^5 g$ in the Adams spectral sequence of *tmf*, derived from power operations (Bruner–Rognes [5]).

Detailed descriptions of these data are available in [12]. Lin's program extends these results by computing additional Adams differentials using tools such as the Leibniz rule, naturality, the Generalized Leibniz Rule, and the Generalized Mahowald Trick. All computed differentials and extensions are accessible via interactive plots [11].

Moreover, the proofs provided in [12] offer more information than the interactive plots. These proofs include numerous disproofs of potential differentials, even for cases where the differentials remain unresolved. For example, consider

$$x_{126,21} \in E_2^{21,126+21}(S^0).$$

The spectral sequence plot for S^0 shows that $x_{126,21}$ survives to the E_4-page, but the value of $d_4(x_{126,21})$ undetermined. By analyzing the proofs in [12], we observe that many potential values for $d_4(x_{126,21})$ have been ruled out. Consequently, we conclude:

$$d_4(x_{126,21}) = x_{125,25,2} + x_{125,25} + g^4 \Delta h_1 g + \text{possibly } d_0^2 e_0 g B_4.$$

Tables 2–12 present results from Lin's program for the classical Adams spectral sequence of the sphere in the range $122 \leq t-s \leq 127, s \leq 25$.

s	Elements	d_r	value
25	$e_0g^3\Delta h_1g$	d_2^{-1}	$g^2\Delta^2m$
	$d_0^3g[B_4]$	d_4	$d_0^4x_{65,13}$
24	$d_0g\Delta^2g^2$	d_2	$d_0e_0g^3\Delta h_2^2$
23	$Ph_1x_{113,18,2}$	d_3^{-1}	$x_{123,20}$
	$h_0^2d_0x_{108,17}$	d_3	d_0^4Mg
22	e_0g^2Mg	d_3^{-1}	$h_0^2h_3x_{116,16}$
	$x_{122,22}$	d_3	$h_0^2d_0x_{107,19}$
	$h_0d_0x_{108,17}$	d_2	$h_0^2d_0x_{107,18}$
21	$h_0^2d_0e_0x_{91,11}$		Permanent
	$d_0x_{108,17}$	d_2	$d_0e_0\Delta h_2^2[B_4] + h_0d_0x_{107,18}$
20	$h_0^4x_{122,16}$	d_2^{-1}	$h_0h_3x_{116,16}$
	$g^3(C_0 + h_0^6h_5^2)$	d_3	$g^3\Delta h_2^2n$
	$h_0d_0e_0x_{91,11}$	d_2	$h_0^6x_{121,16}$
19	$h_0^3x_{122,16}$	d_2^{-1}	$h_3x_{116,16}$
	$d_0e_0x_{91,11}$	d_2	$h_0d_0^2x_{93,12}$
18	$h_0^2x_{122,16}$	d_3^{-1}	$h_0^2x_{123,13,2}$
	$h_1x_{121,17}$	d_7^{-1}	$x_{123,11,2} + x_{123,11} + h_0h_6[B_4]$
	$d_0x_{108,14}$	d_3	$h_0d_0^2x_{93,12} + h_0^5x_{121,16}$
17	$h_0x_{122,16}$	d_3^{-1}	$h_0x_{123,13,2}$
	$g^3[H_1]$	d_3^{-1}	$\Delta h_2^2x_{93,8}$
16	$x_{122,16} + h_0x_{122,15,2}$	d_3^{-1}	$x_{123,13,2}$
	$\Delta h_2^2x_{92,10}$	d_3^{-1}	$x_{123,13}$
	$h_0x_{122,15,2}$		Permanent
15	$x_{122,15}$	d_3	g^3A
	$x_{122,15,2}$	d_2	$h_0^2h_4x_{106,14}$
14	$h_0x_{122,13}$	d_3^{-1}	$x_{123,11,2}$
13	$h_0^2h_6Md_0$	d_2^{-1}	$x_{123,11}$
	$h_1^2x_{120,11}$		Permanent
	$x_{122,13}$	d_3	$h_0h_4x_{106,14}$
12	$h_0h_6Md_0$	d_2^{-1}	$x_{123,10}$
	$h_0x_{122,11}$	d_3^{-1}	$h_0x_{123,8}$
	$h_5x_{91,11}$		Permanent
11	$x_{122,11} + h_6Md_0$	d_3^{-1}	$x_{123,8}$
	h_6Md_0		Permanent
9-10			
8	$h_1x_{121,7}$	d_6	?
0-7			

TABLE 2. The classical Adams spectral sequence of S^0 for $s \leq 25$ in stem 122

s	Elements	d_r	value
25			
24	$h_1Ph_1x_{113,18,2}$	d_2^{-1}	$x_{124,22}$
	$d_0^2\Delta h_2^2Mg$	d_2^{-1}	$d_0x_{110,18}$
	$d_0MPx_{56,10}$	d_4	$MPx_{69,18}$
23	$g^2\Delta^2m$	d_2	$e_0g^3\Delta h_1g$
21-22			
20	$x_{123,20}$	d_3	$Ph_1x_{113,18,2}$
19	$e_0g^2x_{66,7}+h_0^6x_{123,13,2}$	d_2^{-1}	$x_{124,17}$
	$h_0^6x_{123,13,2}$	d_2^{-1}	$h_0^3x_{124,14,2}$
	$h_0e_0x_{106,14}+h_0^2h_3x_{116,16}$	d_3^{-1}	$e_0x_{107,12}$
	$h_0^2h_3x_{116,16}$	d_3	e_0g^2Mg
18	$h_0^5x_{123,13,2}$	d_2^{-1}	$h_0^2x_{124,14,2}$
	$e_0x_{106,14}+h_0h_3x_{116,16}$		Permanent
	$h_0h_3x_{116,16}$	d_2	$h_0^4x_{122,16}$
17	$h_0^2x_{123,15}+h_0^4x_{123,13,2}$	d_2^{-1}	$h_0x_{124,14}$
	$h_0^4x_{123,13,2}$	d_2^{-1}	$h_0x_{124,14,2}+h_0x_{124,14}$
	$d_0x_{109,13}$		Permanent
	$h_3x_{116,16}$	d_2	$h_0^3x_{122,16}$
16	$h_0x_{123,15}+h_0^3x_{123,13,2}$	d_2^{-1}	$x_{124,14}$
	$h_0^3x_{123,13,2}$	d_2^{-1}	$x_{124,14,2}+x_{124,14}$
	$h_1x_{122,15,2}$	d_3^{-1}	$h_4x_{109,12}$
15	$x_{123,15}$	d_4^{-1}	$x_{124,11,2}+x_{124,11}$
	$h_4x_{108,14}$	d_5^{-1}	$h_0x_{124,9}$
	$h_0^2x_{123,13,2}$	d_3	$h_0^2x_{122,16}$
14	$h_0^3x_{123,11}$	d_2^{-1}	$h_0x_{124,11}$
	$h_0x_{123,13,2}$	d_3	$h_0x_{122,16}$
	$\Delta h_2^2x_{93,8}$	d_3	$g^3[H_1]$
13	$h_0^2x_{123,11}$	d_2^{-1}	$x_{124,11}$
	$x_{123,13,2}$	d_3	$x_{122,16}+h_0x_{122,15,2}$
	$x_{123,13}$	d_3	$\Delta h_2^2x_{92,10}$
12	$h_0x_{123,11}+h_0^2h_6[B_4]$	d_3^{-1}	$x_{124,9,2}+h_0x_{124,8}$
	$x_{123,12}$	d_3^{-1}	$x_{124,9}+h_0x_{124,8}$
	$h_0^2h_6[B_4]$	d_5^{-1}	h_6A
11	$h_0^2x_{123,9}$	d_2^{-1}	$x_{124,9}$
	$h_5x_{92,10}$	d_5^{-1}	$x_{124,6}$
	$x_{123,11,2}+x_{123,11}+h_0h_6[B_4]$	d_7	$h_1x_{121,17}$
	$x_{123,11}+h_0h_6[B_4]$	d_3	$h_0x_{122,13}$
	$h_0h_6[B_4]$	d_2	$h_0^2h_6Md_0$
10	$h_0x_{123,9}$	d_2^{-1}	$x_{124,8}$
	$x_{123,10}+h_6[B_4]$	d_3^{-1}	$x_{124,7}$
	$h_6[B_4]$	d_2	$h_0h_6Md_0$
9	$x_{123,9}+h_0x_{123,8}$	d_{12}	?
	$h_0x_{123,8}$	d_3	$h_0x_{122,11}+h_0h_6Md_0$
8	$x_{123,8}$	d_3	$x_{122,11}+h_6Md_0$
0-7			

TABLE 3. The classical Adams spectral sequence of S^0 for $s \leq 25$ in stem 123

s	Elements	d_r	value
25	$h_0^{11}x_{124,14,2}$	d_2^{-1}	$h_0^9x_{125,14}$
	$ix_{101,18}$	d_2	$d_0^2\Delta h_2^2x_{65,13}+h_0Pd_0x_{101,18}$
	$d_0e_0\Delta^3h_1g$	d_2	$d_0^2g^3m$
24	$h_0^{10}x_{124,14,2}$	d_2^{-1}	$h_0^8x_{125,14}$
	$h_0^2d_0x_{110,18}$	d_2^{-1}	$h_0e_0x_{108,17}$
23	$h_0^9x_{124,14,2}$	d_2^{-1}	$h_0^7x_{125,14}$
	$d_0g\Delta h_2^2[B_4]+h_0d_0x_{110,18}$	d_2^{-1}	$e_0x_{108,17}$
	$h_0d_0x_{110,18}$	d_3^{-1}	$x_{125,20}$
	$d_0x_{110,19}$		Permanent
22	$h_0^8x_{124,14,2}$	d_2^{-1}	$h_0^6x_{125,14}$
	$g^2\Delta^2t$	d_3^{-1}	$gx_{105,15}$
	$x_{124,22}$	d_2	$h_1Ph_1x_{113,18,2}$
	$d_0x_{110,18}$	d_2	$d_0^2\Delta h_2^2Mg$
21	$h_0^7x_{124,14,2}$	d_2^{-1}	$h_0^5x_{125,14}$
	$h_0d_0^2[\Delta\Delta_1g]$	d_2^{-1}	$d_0gx_{91,11}$
20	$h_0^6x_{124,14,2}$	d_2^{-1}	$h_0^4x_{125,14}$
	$d_0^2[\Delta\Delta_1g]$	d_5^{-1}	$h_1x_{124,14}$
19	$h_0^5x_{124,14,2}$	d_2^{-1}	$h_0^3x_{125,14}$
	$d_0x_{110,15}$		Permanent
18	$h_0x_{124,17}+h_0^4x_{124,14,2}$	d_2^{-1}	$x_{125,16}$
	$h_0^4x_{124,14,2}$	d_2^{-1}	$h_0^2x_{125,14}$
17	$h_0^3x_{124,14}$	d_2^{-1}	$x_{125,15}$
	$h_0^2x_{124,15}$	d_2^{-1}	$h_0x_{125,14}$
	$x_{124,17}$	d_2	$e_0g^2x_{66,7}+h_0^6x_{123,13,2}$
	$h_0^3x_{124,14,2}$	d_2	$h_0^6x_{123,13,2}$
16	$h_0x_{124,15}$	d_2^{-1}	$x_{125,14}$
	$h_1x_{123,15}$	d_3^{-1}	$h_3x_{118,12}$
	$h_0^2x_{124,14}$		Permanent
	$e_0x_{107,12}$	d_3	$e_0g^2x_{66,7}+h_0e_0x_{106,14}+h_0^2h_3x_{116,16}$
	$h_0^2x_{124,14,2}$	d_2	$h_0^5x_{123,13,2}$
15	$x_{124,15}$	d_4^{-1}	$h_6x_{62,10}$
	$h_3^2x_{110,13}+h_0x_{124,14}$		Permanent
	$h_0x_{124,14}$	d_2	$h_0^2x_{123,15}+h_0^4x_{123,13,2}$
	$h_0x_{124,14,2}$	d_2	$h_0^2x_{123,15}$
14	$h_1x_{123,13}$	d_2^{-1}	$x_{125,12}$
	$h_1x_{123,13,2}$	d_2^{-1}	$x_{125,12,2}$
	$\Delta h_2^2x_{94,8}$		Permanent
	$x_{124,14}$	d_2	$h_0x_{123,15}+h_0^3x_{123,13,2}$
	$x_{124,14,2}$	d_2	$h_0x_{123,15}$
13	$h_0^5x_{124,8}$	d_2^{-1}	$h_0^3x_{125,8}$
	$[H_1](\Delta e_1+C_0+h_0^6h_5^2)$	d_3^{-1}	$x_{125,10,2}$
	$e_0\Delta h_6g$		Permanent
	$h_4x_{109,12}$	d_3	$h_1x_{122,15,2}$

TABLE 4. The classical Adams spectral sequence of S^0 for $13 \leq s \leq 25$ in stem 124

12	$h_0x_{124,11,2}+h_0x_{124,11}$	d_2^{-1}	$x_{125,10}$
	$h_0^2x_{124,10,2}+h_0^4x_{124,8}$	d_2^{-1}	$h_0x_{125,9,2}$
	$h_0^4x_{124,8}$	d_2^{-1}	$h_0^2x_{125,8}$
	$h_1x_{123,11,2}$	d_3^{-1}	$x_{125,9}$
	$h_0x_{124,11}$	d_2	$h_0^3x_{123,11}$
11	$h_0x_{124,10,2}+h_0^3x_{124,8}$	d_2^{-1}	$x_{125,9,2}$
	$h_0^3x_{124,8}$	d_2^{-1}	$h_0x_{125,8}$
	$x_{124,11,3}$	d_3^{-1}	$x_{125,8,2}$
	$x_{124,11,2}+x_{124,11}$	d_4	$x_{123,15}$
	$x_{124,11}$	d_2	$h_0^2x_{123,11}$
10	$h_1x_{123,9}+h_0^2x_{124,8}$	d_2^{-1}	$x_{125,8}$
	$x_{124,10,2}+h_0x_{124,9}$	d_4^{-1}	$h_6[H_1]$
	$x_{124,10}+h_0^2x_{124,8}$	d_4^{-1}	$h_6[H_1]+h_0x_{125,5}$
	$h_0^2x_{124,8}$		Permanent
	$h_0x_{124,9}$	d_5	$h_4x_{108,14}$
9	$x_{124,9,2}+h_0x_{124,8}$	d_3	$h_0x_{123,11}+h_0^2h_6[B_4]$
	$x_{124,9}+h_0x_{124,8}$	d_3	$x_{123,12}$
	$h_0x_{124,8}$	d_2	$h_0^2x_{123,9}$
8	$x_{124,8}$	d_2	$h_0x_{123,9}$
7	$h_0x_{124,6}$	d_2^{-1}	$x_{125,5}$
	h_6A	d_5	$h_0^2h_6[B_4]$
	$x_{124,7}$	d_3	$x_{123,10}+h_6[B_4]$
6	$x_{124,6}$	d_5	$h_5x_{92,10}$
0-5			

TABLE 5. The classical Adams spectral sequence of S^0 for $s \leq 12$ in stem 124

s	Elements	d_r	value
25	$d_0^2e_0g[B_4]$	d_4^{-1}	$h_1x_{125,20}$
	$x_{125,25,2}+x_{125,25}+g^4\Delta h_1g$	d_4^{-1}	$x_{126,21}$ + possibly $h_1x_{125,20}$
	$g^4\Delta h_1g$		Permanent
	$x_{125,25}$	d_2	$h_0x_{124,26}$
24	$e_0g\Delta^2g^2$	d_2	$d_0g^4\Delta h_2^2$
23	$h_0^2e_0x_{108,17}$	d_3	$d_0^3e_0Mg$
	$h_0^9x_{125,14}$	d_2	$h_0^{11}x_{124,14,2}$
22	g^3Mg	d_4^{-1}	$x_{126,18}$
	$ix_{102,15}+h_0^8x_{125,14}$	d_4^{-1}	$gx_{106,14}+e_0x_{109,14,2}$
	$h_0^8x_{125,14}$	d_2	$h_0^{10}x_{124,14,2}$
	$h_0e_0x_{108,17}$	d_2	$h_0^2d_0x_{110,18}$
21	$x_{125,21}$	d_4^{-1}	$d_0x_{112,13}$
	$h_0^7x_{125,14}$	d_2	$h_0^9x_{124,14,2}$
	$e_0x_{108,17}$	d_2	$d_0g\Delta h_2^2[B_4]+h_0d_0x_{110,18}$
20	$h_0d_0gx_{91,11}$	d_2^{-1}	$x_{126,18,2}$
	$x_{125,20}$	d_3	$d_0g\Delta h_2^2[B_4]$
	$h_0^6x_{125,14}$	d_2	$h_0^8x_{124,14,2}$

TABLE 6. The classical Adams spectral sequence of S^0 for $20 \leq s \leq 25$ in stem 125

19	$gx_{105,15}$	d_3	$g^2\Delta^2 t$
	$h_0^5 x_{125,14}$	d_2	$h_0^7 x_{124,14,2}$
	$d_0 g x_{91,11}$	d_2	$h_0 d_0^2[\Delta\Delta_1 g]$
18	$d_0^2 x_{97,10}$	d_5^{-1}	$h_1 x_{125,12,2}$
	$h_0^4 x_{125,14}$	d_2	$h_0^6 x_{124,14,2}$
17	$h_0^2 Q_2 x_{68,8}$	d_2^{-1}	$h_0 D_2 x_{68,8}$
	$h_0^3 x_{125,14}$	d_2	$h_0^5 x_{124,14,2}$
16	$h_0 Q_2 x_{68,8}$	d_2^{-1}	$D_2 x_{68,8}$
	$h_1 x_{124,15}$	d_4^{-1}	$h_0^2 x_{126,10}$
	$x_{125,16}$	d_2	$h_0 x_{124,17} + h_0^4 x_{124,14,2}$
	$h_0^2 x_{125,14}$	d_2	$h_0^4 x_{124,14,2}$
15	$h_0^3 x_{125,12}$	d_2^{-1}	$h_0 h_3 x_{119,11}$
	$Q_2 x_{68,8}$	d_4^{-1}	$h_1 x_{125,10}$
	$h_1 x_{124,14}$	d_5	$d_0^2[\Delta\Delta_1 g]$
	$x_{125,15}$	d_2	$h_0^3 x_{124,14}$
	$h_0 x_{125,14}$	d_2	$h_0^2 x_{124,15}$
14	$h_0^2 x_{125,12}$	d_2^{-1}	$h_3 x_{119,11}$
	$h_1 h_4 x_{109,12}$		Permanent
	$x_{125,14}$	d_2	$h_0 x_{124,15}$
13	$h_0^4 x_{125,9,2}$	d_2^{-1}	$x_{126,11}$
	$h_0^5 x_{125,8}$	d_2^{-1}	$h_0 x_{126,10}$
	$n x_{94,8}$	d_4^{-1}	$h_1 x_{125,8}$
	$h_0 x_{125,12}$	d_4^{-1}	$h_1 x_{125,8,2}$
	$h_3 x_{118,12}$	d_3	$h_1 x_{123,15}$
12	$h_0 h_6 x_{62,10}$	d_2^{-1}	$x_{126,10}$
	$h_0^3 x_{125,9,2} + h_0^4 x_{125,8}$	d_3^{-1}	$x_{126,9}$
	$h_0^4 x_{125,8}$	d_3^{-1}	$x_{126,9} + h_0 x_{126,8,3}$
	$x_{125,12}$	d_2	$h_1 x_{123,13}$
	$x_{125,12,2}$	d_2	$h_1 x_{123,13,2}$
11	$h_1 x_{124,10,2}$	d_3^{-1}	$x_{126,8}$
	$h_1 x_{124,10}$	d_3^{-1}	$x_{126,8,2}$
	$h_0^2 x_{125,9,2}$	d_5	?
	$h_6 x_{62,10}$	d_4	$x_{124,15}$
	$h_0^3 x_{125,8}$	d_2	$h_0^5 x_{124,8}$
10	$h_0 x_{125,9}$	d_2^{-1}	$x_{126,8,3}$
	$x_{125,10,2}$	d_3	$[H_1](\Delta e_1 + C_0 + h_0^6 h_5^2)$
	$x_{125,10}$	d_2	$h_0 x_{124,11,2} + h_0 x_{124,11}$
	$h_0 x_{125,9,2}$	d_2	$h_0^2 x_{124,10,2} + h_0^4 x_{124,8}$
	$h_0^2 x_{125,8}$	d_2	$h_0^4 x_{124,8}$
9	$h_6(\Delta e_1 + C_0 + h_0^6 h_5^2)$		Permanent
	$h_5 x_{94,8}$	d_7	?
	$x_{125,9}$	d_3	$h_1 x_{123,11,2}$
	$x_{125,9,2}$	d_2	$h_0 x_{124,10,2} + h_0^3 x_{124,8}$
	$h_0 x_{125,8}$	d_2	$h_0^3 x_{124,8}$
8	$x_{125,8,2}$	d_3	$x_{124,11,3}$
	$x_{125,8}$	d_2	$h_1 x_{123,9} + h_0^2 x_{124,8}$
7	$h_0^2 x_{125,5}$	d_3^{-1}	$x_{126,4}$
6	$h_6[H_1]$	d_4	$x_{124,10,2} + h_0 x_{124,9}$
	$h_0 x_{125,5}$	d_4	$x_{124,10,2} + x_{124,10} + h_0 x_{124,9} + h_0^2 x_{124,8}$
5	$x_{125,5}$	d_2	$h_0 x_{124,6}$
0-4			

TABLE 7. The classical Adams spectral sequence of S^0 for $s \leq 19$ in stem 125

s	Elements	d_r	value
25	$h_0^7x_{126,18}$	d_3^{-1}	$h_0^{21}h_7$
24	$d_0e_0\Delta h_2^2Mg$	d_2^{-1}	$d_0x_{113,18}$
	$h_0^6x_{126,18}$	d_3^{-1}	$h_0^{20}h_7$
	$g^4\Delta h_2c_1$	d_3^{-1}	g^3C''
	$d_0Pd_0M^2$	d_4^{-1}	$d_0e_0[\Delta\Delta_1g]$
23	$h_0^5x_{126,18}$	d_3^{-1}	$h_0^{19}h_7$
	$x_{126,23}$	d_4^{-1}	$e_0x_{110,15}$
22	$h_0x_{126,21}+h_0^4x_{126,18}$	d_3^{-1}	$h_1x_{126,18,2}$
	$h_0^4x_{126,18}$	d_3^{-1}	$h_0^{18}h_7$
21	$h_0^3x_{126,18}$	d_3^{-1}	$h_0^{17}h_7$
	$h_1x_{125,20}$	d_4	$d_0^2e_0g[B_4]$
	$x_{126,21}$	d_4	$x_{125,25,2}+x_{125,25}+g^4\Delta h_1g+$ possibly $d_0^2e_0gB_4$
20	$h_0^2x_{126,18}$	d_3^{-1}	$h_0^{16}h_7$
	$d_0x_{112,16}$	d_5^{-1}	$x_{127,15}$
19	$h_0x_{126,18}$	d_3^{-1}	$h_0^{15}h_7$
	$g^3x_{66,7}$	d_3^{-1}	$gx_{107,12}$
18	$x_{126,18}+e_0x_{109,14,2}$	d_7	?
	$e_0x_{109,14,2}$	d_4	g^3Mg
	$gx_{106,14}$	d_4	$ix_{102,15}+g^3Mg+h_0^8x_{125,14}$
	$x_{126,18,2}$	d_2	$h_0d_0gx_{91,11}$
17	$h_0^{15}h_6^2$	d_2^{-1}	$h_0^{14}h_7$
	$h_1^2x_{124,15}$	d_2^{-1}	$h_6x_{64,14}$
	$x_{126,17}$	d_8	?
	$d_0x_{112,13}$	d_4	$x_{125,21}$
16	$h_0^{14}h_6^2$	d_2^{-1}	$h_0^{13}h_7$
	$h_0^2D_2x_{68,8}$	d_3^{-1}	$x_{127,13}$
	$h_1^2x_{124,14}$	d_6^{-1}	$h_2x_{124,9}+h_0^2x_{127,8}$
15	$h_0^{13}h_6^2$	d_2^{-1}	$h_0^{12}h_7$
	$h_0D_2x_{68,8}$	d_2	$h_0^2Q_2x_{68,8}$
14	$h_0^{12}h_6^2$	d_2^{-1}	$h_0^{11}h_7$
	$x_{126,14}$	d_4^{-1}	$h_0^2x_{127,8}$
	$h_1h_3x_{118,12}$	d_5^{-1}	$h_1x_{126,8,2}$
	$D_2x_{68,8}$	d_2	$h_0Q_2x_{68,8}$
13	$h_0^{11}h_6^2$	d_2^{-1}	$h_0^{10}h_7$
	$h_1x_{125,12,2}$	d_5	$d_0^2x_{97,10}$
	$h_0h_3x_{119,11}$	d_2	$h_0^3x_{125,12}$
12	$d_1x_{94,8}$	d_2^{-1}	$x_{127,10}$
	$h_0x_{126,11}$	d_2^{-1}	$h_3x_{120,9}$
	$h_0^{10}h_6^2$	d_2^{-1}	$h_0^9h_7$
	$h_0^2x_{126,10}$	d_4	$h_1x_{124,15}$
	$h_3x_{119,11}$	d_2	$h_0^2x_{125,12}$
11	$h_0^2x_{126,9}$	d_2^{-1}	$h_0x_{127,8}$
	$h_0^9h_6^2$	d_2^{-1}	$h_0^8h_7$
	$h_1x_{125,10,2}+h_1x_{125,10}$		Permanent
	$h_1x_{125,10}$	d_4	$Q_2x_{68,8}$
	$x_{126,11}$	d_2	$h_0^4x_{125,9,2}$
	$h_0x_{126,10}$	d_2	$h_0^5x_{125,8}$

TABLE 8. The classical Adams spectral sequence of S^0 for $11 \leq s \leq 25$ in stem 126

10	$h_0x_{126,9}$	d_2^{-1}	$x_{127,8}$
	$h_0^2x_{126,8}$	d_2^{-1}	$h_0x_{127,7}$
	$h_0^8h_6^2$	d_2^{-1}	$h_0^7h_7$
	$h_0^2x_{126,8,3}$		Permanent
	$x_{126,10}$	d_2	$h_0h_6x_{62,10}$
9	$h_0x_{126,8}$	d_2^{-1}	$x_{127,7}$
	$h_0^7h_6^2$	d_2^{-1}	$h_0^6h_7$
	$h_1x_{125,8}$	d_4	$nx_{94,8}$
	$h_1x_{125,8,2}$	d_4	$h_0x_{125,12}$
	$x_{126,9}$	d_3	$h_0^3x_{125,9,2} + h_0^4x_{125,8}$
	$h_0x_{126,8,3}$	d_3	$h_0^3x_{125,9,2}$
8	$h_0^6h_6^2$	d_2^{-1}	$h_0^5h_7$
	$h_6(C' + X_2)$	d_{17}	?
	$x_{126,8,4} + x_{126,8}$	d_6	?
	$x_{126,8}$	d_3	$h_1x_{124,10,2}$
	$x_{126,8,2}$	d_3	$h_1x_{124,10}$
	$x_{126,8,3}$	d_2	$h_0x_{125,9}$
7	$h_0^5h_6^2$	d_2^{-1}	$h_0^4h_7$
	$h_1h_6[H_1]$	d_{18}	?
6	$h_0^4h_6^2$	d_2^{-1}	$h_0^3h_7$
	$x_{126,6}$	d_3	$h_5x_{94,8}$ + possibly $h_6(\Delta e_1 + C_0 + h_0^6h_5^2)$
5	$h_0^3h_6^2$	d_2^{-1}	$h_0^2h_7$
4	$h_0^2h_6^2$	d_2^{-1}	h_0h_7
	$x_{126,4}$	d_3	$h_0^2x_{125,5}$
3	$h_0h_6^2$	d_2^{-1}	h_7
2	h_6^2	d_7	?
0-1			

TABLE 9. The classical Adams spectral sequence of S^0 for $s \leq 10$ in stem 126

s	Elements	d_r	value
25	$h_0^{24}h_7$	d_3	$h_0^{10}x_{126,18}$
	$ix_{104,18}$	d_2	$d_0^3x_{84,15,2} + h_0d_0x_{112,22}$
	$d_0g\Delta^3h_1g$	d_2	$d_0e_0g^3m$
24	$h_0^2d_0x_{113,18}$	d_2^{-1}	$h_0gx_{108,17}$
	$h_1x_{126,23}$	d_3^{-1}	$x_{128,21}$
	$h_0^{23}h_7$	d_3	$h_0^9x_{126,18}$
23	$e_0g\Delta h_2^2[B_4] + h_0d_0x_{113,18}$	d_2^{-1}	$gx_{108,17}$
	$h_0d_0x_{113,18}$	d_4	$d_0^3x_{84,15,2}$
	$h_0^{22}h_7$	d_3	$h_0^8x_{126,18}$
	$d_0Pd_0x_{91,11}$	d_3	$h_1x_{125,25}$
22	$d_0x_{113,18,2}$	d_4^{-1}	$d_0e_0x_{97,10}$
	$h_0^{21}h_7$	d_3	$h_0^7x_{126,18}$
	$d_0x_{113,18}$	d_2	$d_0e_0\Delta h_2^2Mg$
21	$h_0^6x_{127,15}$	d_2^{-1}	$h_0^5x_{128,14}$
	$x_{127,21} + g^3C''$		Permanent
	$h_0^{20}h_7$	d_3	$h_0^6x_{126,18}$
	g^3C''	d_3	$g^4\Delta h_2c_1$

TABLE 10. The classical Adams spectral sequence of S^0 for $21 \leq s \leq 25$ in stem 127

20	$h_0^5 x_{127,15}$	d_2^{-1}	$h_0^4 x_{128,14}$
	$d_0 e_0 [\Delta\Delta_1 g]$	d_4	$d_0 P d_0 M^2$
	$h_0^{19} h_7$	d_3	$h_0^5 x_{126,18}$
19	$h_0^4 x_{127,15}$	d_2^{-1}	$h_0^3 x_{128,14}$
	$h_1 x_{126,18}$		Permanent
	$e_0 x_{110,15}$	d_4	$x_{126,23}$
	$h_1 x_{126,18,2}$	d_3	$h_0 x_{126,21} + h_0^4 x_{126,18}$
	$h_0^{18} h_7$	d_3	$h_0^4 x_{126,18}$
18	$h_0^3 x_{127,15}$	d_2^{-1}	$h_0^2 x_{128,14}$
	$h_0^3 h_6 x_{64,14}$	d_2^{-1}	$h_0^2 h_6 x_{65,13}$
	$g^2 \Delta h_1 H_1$	d_3^{-1}	$g x_{108,11}$
	$h_1 x_{126,17}$		Permanent
	$h_0^{17} h_7$	d_3	$h_0^3 x_{126,18}$
17	$h_0^2 h_2 x_{124,14}$	d_2^{-1}	$x_{128,15}$
	$h_0^2 x_{127,15}$	d_2^{-1}	$x_{128,15} + h_0 x_{128,14}$
	$h_0^2 h_6 x_{64,14}$	d_2^{-1}	$h_0 h_6 x_{65,13}$
	$g x_{107,13}$	d_4^{-1}	$h_0 h_3 x_{121,11}$
	$h_0^{16} h_7$	d_3	$h_0^2 x_{126,18}$
16	$h_0 x_{127,15} + h_0 h_2 x_{124,14}$	d_2^{-1}	$x_{128,14}$
	$h_0 h_6 x_{64,14}$	d_2^{-1}	$h_6 x_{65,13}$
	$h_0 h_2 x_{124,14}$		Permanent
	$x_{127,16}$		Permanent
	$h_0^{15} h_7$	d_3	$h_0 x_{126,18}$
	$g x_{107,12}$	d_3	$g^3 x_{66,7}$
15	$h_1 x_{126,14}$	d_2^{-1}	$x_{128,13,2}$
	$h_2 x_{124,14}$		Permanent
	$x_{127,15}$	d_5	$d_0 x_{112,16}$
	$h_0^{14} h_7$	d_2	$h_0^{15} h_6^2$
	$h_6 x_{64,14}$	d_2	$h_1^2 x_{124,15}$
14	$h_0 g \Delta h_6 g$	d_2^{-1}	$x_{128,12,2}$
	$h_0 h_3 x_{120,12}$	d_2^{-1}	$h_3 x_{121,11}$
	$h_0^{13} h_7$	d_2	$h_0^{14} h_6^2$
13	$h_0^3 x_{127,10}$	d_2^{-1}	$h_0 x_{128,10}$
	$g \Delta h_6 g$	d_3^{-1}	$x_{128,10,2}$
	$h_3 x_{120,12}$	d_4^{-1}	$h_2 x_{125,8,2}$
	$x_{127,13}$	d_3	$h_0^2 D_2 x_{68,8}$
	$h_0^{12} h_7$	d_2	$h_0^{13} h_6^2$
12	$h_0^2 x_{127,10}$	d_2^{-1}	$x_{128,10}$
	$h_1 x_{126,11}$	d_3^{-1}	$h_1 x_{127,8}$
	$h_0^{11} h_7$	d_2	$h_0^{12} h_6^2$
11	$h_0 h_3 x_{120,9}$	d_3^{-1}	$h_3 D_2 h_6$
	$h_0 h_2 x_{124,9}$		Permanent
	$h_0 x_{127,10}$		Permanent
	$h_0^{10} h_7$	d_2	$h_0^{11} h_6^2$
10	$h_1^2 x_{125,8}$		Permanent
	$h_2 x_{124,9} + h_0^2 x_{127,8}$	d_6	$h_1^2 x_{124,14}$
	$h_0^2 x_{127,8}$	d_4	$x_{126,14}$
	$x_{127,10}$	d_2	$d_1 x_{94,8}$
	$h_3 x_{120,9}$	d_2	$h_0 x_{126,11}$
	$h_0^9 h_7$	d_2	$h_0^{10} h_6^2$

TABLE 11. The classical Adams spectral sequence of S^0 for $10 \leq s \leq 20$ in stem 127

9	$h_0^2x_{127,7}$	d_2^{-1}	$h_0x_{128,6}$
	$h_1x_{126,8}$		Permanent
	$h_1x_{126,8,2}$	d_5	$h_1h_3x_{118,12}$
	$h_0x_{127,8}$	d_2	$h_0^2x_{126,9}$
	$h_0^8h_7$	d_2	$h_0^9h_6^2$
8	$h_0x_{127,7,2}+h_0x_{127,7}+h_0^2x_{127,6}$	d_2^{-1}	$x_{128,6}$
	$h_0^2x_{127,6}$	d_2^{-1}	$h_0x_{128,5}$
	h_2h_6A		Permanent
	$h_2x_{124,7}$	d_9	?
	$x_{127,8}$	d_2	$h_0x_{126,9}$
	$h_0x_{127,7}$	d_2	$h_0^2x_{126,8}$
	$h_0^7h_7$	d_2	$h_0^8h_6^2$
7	$h_0x_{127,6}$	d_2^{-1}	$x_{128,5}$
	$h_1x_{126,6}$	d_{10}	?
	$x_{127,7,2}+x_{127,7}$	d_3	?
	$x_{127,7}$	d_2	$h_0x_{126,8}$
	$h_0^6h_7$	d_2	$h_0^7h_6^2$
6	$x_{127,6}$	d_4	?
	$h_0^5h_7$	d_2	$h_0^6h_6^2$
5	$h_0^4h_7$	d_2	$h_0^5h_6^2$
4	$h_0^3h_7$	d_2	$h_0^4h_6^2$
3	$h_1h_6^2$	d_{14}	?
	$h_0^2h_7$	d_2	$h_0^3h_6^2$
2	h_0h_7	d_2	$h_0^2h_6^2$
1	h_7	d_2	$h_0h_6^2$
0			

TABLE 12. The classical Adams spectral sequence of S^0 for $s \leq 9$ in stem 127

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